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High order Fuchsian equations for the square lattice Ising model: $\tilde{\chi}^{(5)}$

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Abstract
We consider the Fuchsian linear differential equation obtained (modulo a prime) for $\tilde{\chi}^{(5)}$, the five-particle contribution to the susceptibility of the square lattice Ising model. We show that one can understand the factorization of the corresponding linear differential operator from calculations using just a single prime. A particular linear combination of $\tilde{\chi}^{(1)}$ and $\tilde{\chi}^{(3)}$ can be removed from $\tilde{\chi}^{(5)}$ and the resulting series is annihilated by a high order globally nilpotent linear ODE. The corresponding (minimal order) linear differential operator, of order 29, splits into factors of small orders. A fifth-order linear differential operator occurs as the left-most factor of the 'depleted' differential operator and it is shown to be equivalent to the symmetric fourth power of L_E, the linear differential operator corresponding to the elliptic integral E. This result generalizes what we have found for the lower order terms $\tilde{\chi}^{(3)}$ and $\tilde{\chi}^{(4)}$. We conjecture that a linear differential operator equivalent to a symmetric $(n-1)$ th power of L_E occurs as a left-most factor in the minimal order linear differential operators for all $\tilde{\chi}^{(n)}$'s.

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1. Introduction

Wu, McCoy, Tracy and Barouch [1] have shown that the magnetic susceptibility of the square lattice Ising model can be expressed as an infinite sum of contributions, known as *n-particle contributions*, so that the high-temperature susceptibility is given by

$$kT \cdot \chi_H(w) = \sum \chi^{(2n+1)}(w) = \frac{1}{s} \cdot (1-s^4)^{\frac{1}{4}} \cdot \sum \tilde{\chi}^{(2n+1)}(w) \tag{1}$$

and the low-temperature susceptibility is given by

$$kT \cdot \chi_L(w) = \sum \chi^{(2n)}(w) = (1 - 1/s^4)^{\frac{1}{4}} \cdot \sum \tilde{\chi}^{(2n)}(w) \tag{2}$$

in terms of the self-dual temperature variable $w = \frac{1}{2}s/(1+s^2)$, with $s = \sinh(2J/kT)$.

As is now well known [1], the n-particle contributions have an integral representation and are given by the $(n-1)$ dimensional integrals [2–5]

$$\tilde{\chi}^{(n)}(w) = \frac{1}{n!} \cdot \left(\prod_{j=1}^{n-1} \int_0^{2\pi} \frac{\mathrm{d}\phi_j}{2\pi} \right) \left(\prod_{j=1}^{n} y_j \right) \cdot R^{(n)} \cdot (G^{(n)})^2, \tag{3}$$

where[6]

$$G^{(n)} = \prod_{1 \leqslant i < j \leqslant n} h_{ij}, \qquad h_{ij} = \frac{2\sin\left((\phi_i - \phi_j)/2\right) \cdot \sqrt{x_i x_j}}{1 - x_i x_j}, \tag{4}$$

and

$$R^{(n)} = \frac{1 + \prod_{i=1}^{n} x_i}{1 - \prod_{i=1}^{n} x_i}, \tag{5}$$

with

$$x_i = \frac{2w}{1 - 2w\cos(\phi_i) + \sqrt{(1-2w\cos(\phi_i))^2 - 4w^2}}, \tag{6}$$

$$y_i = \frac{2w}{\sqrt{(1-2w\cos(\phi_i))^2 - 4w^2}}, \qquad \sum_{j=1}^{n} \phi_j = 0 \tag{7}$$

valid for small w and, elsewhere, by analytic continuation. The variable w corresponds to small values of s *as well as* large values of s. It is worth noting that the series expansions for $\tilde{\chi}^{(n)}$ in the variable w have integer coefficients. From the first $\tilde{\chi}^{(n)}$, the coefficients for generic n can be inferred [6]

$$\tilde{\chi}^{(n)} = 2^n \cdot w^{n^2} \cdot \big(1 + 4n^2 \cdot w^2 + 2 \cdot (4n^4 + 13n^2 + 1) \cdot w^4 + \tfrac{8}{3} \cdot (n^2+4)(4n^4 + 23n^2 + 3) \cdot w^6 + \cdots\big), \tag{8}$$

where the coefficients are valid up to w^2 for $n \geqslant 3$, w^4 for $n \geqslant 5$ and w^6 for $n \geqslant 7$ (in particular it should be noted that $\tilde{\chi}^{(n)}$ is an even function only for n even).

In previous work [7] we performed massive computer calculations to obtain the susceptibility of the square lattice Ising model and the n-particle contributions $\tilde{\chi}^{(n)}$. These calculations confirmed previously [8, 9] conjectured singularities for the linear ODEs of $\tilde{\chi}^{(n)}$ (for $n = 5$ and 6) and yielded the values of the associated local exponents. In addition some light was shed [7] on important physical problems such as the existence of a *natural boundary* for the susceptibility of the square lattice Ising model and the subtle *resummation of*

[6] The Fermionic term $G^{(n)}$ has several representations [5].

logarithmic contributions from individual $\tilde{\chi}^{(n)}$'s resulting in the power-law behaviour of the full susceptibility χ.

As far as the five-particle contribution to the susceptibility is concerned, a long series $S(w)$ for $\tilde{\chi}^{(5)}$ was generated modulo the prime $p_r = 2^{15} - 19$ from which we obtained [7] the corresponding Fuchsian differential equation. This Fuchsian linear ODE is of order 33 and we denote by L_{33} its linear differential operator. The calculation of the series is very time consuming and one cannot calculate (given presently available computational resources) the many series modulo various primes required to reconstruct, through the *Chinese remainder* procedure, the exact series for $\tilde{\chi}^{(5)}$, and, from this, deduce the corresponding exact Fuchsian linear ODE. Our purpose here is, using the series and the linear ODE obtained modulo a single prime, to perform, as far as possible, the factorization of the linear differential operator L_{33} and gain as deep an understanding as possible of the various factors occurring in its exact factorization (over the rationals).

In particular, we find that a certain linear combination of $\tilde{\chi}^{(1)}$ and $\tilde{\chi}^{(3)}$ can be removed from $\tilde{\chi}^{(5)}$ and the resulting series is a solution of an order 29 linear ODE. We develop methods which enable us to show that the corresponding linear differential operator L_{29} splits into several factors and we present arguments that the order of any individual factor may not exceed five. The factor L_5 of maximum order occurs as the left-most factor of L_{29}. We show that L_5 is equivalent[7] to the symmetric fourth power of L_E, the linear differential operator corresponding to the complete elliptic integral E, see (53). This result generalizes what we have found in [10–12] for the lower terms $\tilde{\chi}^{(3)}$ and $\tilde{\chi}^{(4)}$. We therefore conjecture that a linear differential operator L_n, equivalent to the symmetric $(n-1)$ th power of L_E, occurs as the left-most factor in the (minimal order) linear differential operators for all $\tilde{\chi}^{(n)}$'s.

2. Deciphering the structure of $\tilde{\chi}^{(n)}$: direct sums, symmetric powers and modular forms

A linear differential operator L can be viewed formally as a non-commutative polynomial in w and D_w, where $D_w = \mathrm{d}/\mathrm{d}w$ is the derivation (or derivative) with respect to w. In previous works [10–12] we have shown that the (minimal order) linear differential operators for $\tilde{\chi}^{(3)}$ and $\tilde{\chi}^{(4)}$ (called respectively L_7 and L_{10}) have a 'Russian-doll' structure involving the differential operators L_1 and N_0 for $\tilde{\chi}^{(1)}$ and $\tilde{\chi}^{(2)}$, respectively. More precisely, $\tilde{\chi}^{(1)}$ and $\tilde{\chi}^{(2)}$ are solutions of the linear ODEs corresponding to L_7 and L_{10}, respectively. In terms of linear differential operators this means that L_1 (respectively N_0) right divides L_7 (respectively L_{10}). Note that throughout this paper when we talk about a homogeneous linear differential equation and its associated differential operator we will use the terms ODE and differential operator, interchangeably.

One might then conjecture that this structure extends to the linear differential operator L_{33} (for $\tilde{\chi}^{(5)}$) and the linear differential operator L_7 (for $\tilde{\chi}^{(3)}$). We note that the singularities for the ODEs corresponding to $\tilde{\chi}^{(3)}$ and $\tilde{\chi}^{(5)}$ are consistent with this assumption, that is *all* the singularities of L_7 also occur in L_{33}. The check of the right division between operators can be done simply by generating the series $L_7\,(S(w))$ and obtaining the corresponding linear ODE. If the order of this latter linear ODE is less than 33, the assumption is verified, i.e. L_7 right divides L_{33}. As reported in [7] this procedure leads to the factorization

$$L_{33} = N_{26} \cdot L_7, \qquad (9)$$

where N_{26} is a linear differential operator of order 26.

A stronger conjecture amounts to saying that the linear differential operator for $\tilde{\chi}^{(n)}$ occurs as part of a *direct-sum decomposition* of the linear differential operator for $\tilde{\chi}^{(n+2)}$. This

[7] For the notion of equivalence of linear differential operators see [18, 19].

conjecture is based on our findings [10–12] that the linear combinations $6\tilde{\chi}^{(n+2)} - n\tilde{\chi}^{(n)}$ with $n = 1, 2$, happen to have a linear ODE of lower order than that for $\tilde{\chi}^{(n+2)}$. The conjecture was verified for $n = 3$ by obtaining [7] the minimal order Fuchsian linear ODE for $6\tilde{\chi}^{(5)} - 3\tilde{\chi}^{(3)}$, which happens to be of order 30, so that

$$L_{33} = L_7 \oplus L_{30}. \tag{10}$$

This direct-sum structure, where a seventh-order linear differential operator combined with an order 30 linear differential operator gives rise to an order 33 linear differential operator, leads to the conclusion that there must be a fourth-order linear differential operator that right divides both L_{30} and L_7. This kind of direct-sum structure (10) is not seen in L_7 or L_{10}, the linear differential operators for $\tilde{\chi}^{(3)}$ and $\tilde{\chi}^{(4)}$.

Recalling [13] the factorization of L_7, the fourth-order linear differential operator reads (following the notation of equation (7) in [13])

$$B_2 \cdot T_1 \cdot L_1 = B_2 \cdot O_1 \cdot N_1 = X_1 \cdot Z_2 \cdot N_1 = L_1 \oplus (Z_2 \cdot N_1) \tag{11}$$

and the factorization of L_{30} in (10) becomes

$$L_{30} = L_{26} \cdot (L_1 \oplus (Z_2 \cdot N_1)). \tag{12}$$

In a further step, Nickel[8] has succeeded in showing that the differential operator L_1 (corresponding to $\tilde{\chi}^{(1)}$) occurs via a direct sum in L_{30}. This was done by considering the series for the combination

$$\tilde{\chi}^{(5)} - \tfrac{1}{2}\tilde{\chi}^{(3)} - \alpha \cdot \tilde{\chi}^{(1)}. \tag{13}$$

If a rational value of α can be found such that the resulting linear ODE has an order less than 30 (in fact equal to 29), then the direct sum assumption has been validated. This happens with $\alpha = -1/120$ and the final result is that the combination

$$\tilde{\chi}^{(5)} - \tfrac{1}{2}\tilde{\chi}^{(3)} + \tfrac{1}{120}\tilde{\chi}^{(1)} \tag{14}$$

is annihilated by an order 29 linear differential operator that we denote L_{29}, leading us to conclude that

$$L_{30} = L_1 \oplus L_{29}. \tag{15}$$

At this point, guided by our results for the three terms $\tilde{\chi}^{(n)}$, $n = 3, 4, 5$, one may wonder whether there is a common structure to the corresponding linear differential operators.

Recall that the ODE for $\tilde{\chi}^{(3)}$ is of order seven and that $\tilde{\chi}^{(3)}$ can be written as

$$\tilde{\chi}^{(3)} = \tfrac{1}{6}\tilde{\chi}^{(1)} + \Phi^{(3)}, \tag{16}$$

where $\Phi^{(3)}$ is a solution of a sixth-order linear ODE. We thus have the *direct-sum decomposition*

$$L_7 = L_1 \oplus L_6. \tag{17}$$

The sixth-order operator L_6 has a third-order linear differential operator Y_3 as a left-most factor

$$L_6 = Y_3 \cdot Z_2 \cdot N_1, \tag{18}$$

and we have given the solutions of the linear ODE corresponding to Y_3 in [13]. These solutions can be written [13] as a homogeneous polynomial of the complete elliptic integrals K and E with homogeneous degree two (the order of Y_3 minus one).

Next we consider the tenth-order linear ODE for $\tilde{\chi}^{(4)}$. Recall that $\tilde{\chi}^{(4)}$ can be written as

$$\tilde{\chi}^{(4)} = \tfrac{1}{3}\tilde{\chi}^{(2)} + \Phi^{(4)}, \tag{19}$$

[8] We are grateful to B G Nickel for this result.

where $\Phi^{(4)}$ is a solution of an eighth-order linear ODE. We thus have the direct-sum decomposition

$$L_{10} = N_0 \oplus L_8. \tag{20}$$

The eighth-order operator L_8 has a fourth-order linear differential operator M_2 as its left-most factor

$$L_8 = M_2 \cdot L_4. \tag{21}$$

The fourth-order linear differential operator L_4 factorizes into four first-order linear differential operators as shown in equation (42) of [11]. As mentioned in [13] (and now given explicitly in appendix A, the linear ODE corresponding to M_2 annihilates a homogeneous polynomial of K and E of (homogeneous) degree three, i.e. the order of M_2 minus one.

Similarly, we have shown that for $\tilde{\chi}^{(5)}$

$$\tilde{\chi}^{(5)} = \tfrac{1}{2}\tilde{\chi}^{(3)} - \tfrac{1}{120}\tilde{\chi}^{(1)} + \Phi^{(5)}, \tag{22}$$

where $\Phi^{(5)}$ is annihilated by an order 29 linear ODE whose corresponding differential operator we denote as L_{29}.

We conjecture that once $\tilde{\chi}^{(n)}$ are depleted of the contributions $\tilde{\chi}^{(n-2k)}$ of lower index (with coefficients α_{n-2k}, where $\alpha_{n-2} = (n-2)/6$, and the remaining coefficients are to be determined numerically)

$$\tilde{\chi}^{(n)} = \alpha_{n-2} \cdot \tilde{\chi}^{(n-2)} + \alpha_{n-4} \cdot \tilde{\chi}^{(n-4)} + \cdots + \Phi^{(n)}, \tag{23}$$

the depleted series $\Phi^{(n)}$ will be solutions of linear ODEs of minimal order q, whose corresponding (minimal order) linear differential operators factorize as

$$L_q = L_n \cdot L_{q-n}, \tag{24}$$

and where the linear ODE corresponding to the left-most factor L_n has as a solution a homogeneous polynomial of complete elliptic integrals E and K of degree $n-1$ (in other words L_n is equivalent to the $(n-1)$ th symmetric power of L_E, annihilating E, see below).

This is what happens for the terms $\tilde{\chi}^{(3)}$ and $\tilde{\chi}^{(4)}$. One of the purposes of this paper is to show that this structure *also holds* for $\tilde{\chi}^{(5)}$. This amounts to demonstrating the occurrence of a fifth-order linear differential operator L_5 at the left of L_{29}, with L_5 being equivalent to the symmetric fourth power of the linear differential operator L_E corresponding to the complete elliptic integrals E (or K).

Before proceeding to show how we achieved this goal, some general remarks are in order. For an integral representation of a function its series expansion $S(x)$ around the origin $(x=0)$ is unique. This series can be annihilated by (is a solution of) many linear ODEs of order Q and degree D (see appendix B)

$$L_{QD} = \sum_{i=0}^{Q} \left(\sum_{j=0}^{D} a_{ij} \cdot x^j \right) \cdot \left(x \frac{\mathrm{d}}{\mathrm{d}x} \right)^i. \tag{25}$$

Among all these linear ODEs there is one of minimal order q and it is unique (its corresponding degree will be denoted by D_0). In terms of linear differential operators, the minimal order differential operator appears as a right factor in the non-minimal order linear differential operators. The minimal order linear ODE may contain a very large number of apparent singularities and can thus only be determined from a very large number of series coefficients (generally speaking $(q+1)(D_0+1)$ terms are needed). Other (non-minimal order) linear ODEs, because they carry polynomials of smaller degrees, may require *fewer series coefficients* in order to be obtained. For any $Q > q$, a linear ODE annihilating $S(x)$

(i.e. $L_{QD}(S(x)) = 0$), can be found[9] for D sufficiently large, and if Q is small enough we can choose Q and D such that $(Q+1)(D+1) < (q+1)(D_0+1)$. Among the non-minimal linear ODEs there will generally be one requiring the *minimal number of terms*; in a computational sense one may view this as the 'optimal linear' ODE[10]. In the case of $\tilde{\chi}^{(5)}$ this optimal linear ODE can be discovered from some 7400 terms while the minimal order linear ODE requires almost 49 100 terms. So when we consider, for instance, a linear differential operator such as L_{29} (of minimal order 29), we are dealing in fact (as far as the computations are concerned) with a much higher order linear differential operator.

Knowledge about the minimal order is 'inferred' from many non-minimal order ODEs by using the remarkable formula (26) below, which we reported in [7]. Seeking a Fuchsian linear ODE of order Q and degree D which annihilates a given series requires a certain number of coefficients N. Formula (26) *relates this number of required coefficients N to the order Q and degree D* of the Fuchsian linear ODE. Remarkably, this 'ODE formula' gives the number of required coefficients N as a linear combination of the order Q and degree D, while a naive and obvious upper bound for N is $(Q+1)(D+1)$. We denote by f the difference between this naive upper bound and the actual *number of required coefficients N*.

We have no proof of this formula, but it has been found to work [7] for all the cases we have considered

$$N = d \cdot Q + q \cdot D - C = (Q+1)(D+1) - f. \qquad (26)$$

This ODE formula is revisited in greater detail in appendix B where all its parameters have now been understood. In all cases we have investigated, the parameter q appearing in (26) is the minimal order of the linear ODE that annihilates $S(x)$. The parameter d is the *number of singularities* (counted with multiplicity) excluding any apparent singularities and the 'true' singular point $x = 0$, which is already taken care of by the use of the (so-called homogeneity) differential operator $x\frac{\mathrm{d}}{\mathrm{d}x}$. Finally we note that the degree of the apparent polynomial of the minimal order linear ODE (as well as the other parameters d, q, C and f) can be extracted exactly without obtaining the minimal order linear ODE (see (B.4) in appendix B).

As stated above we are dealing with linear differential operators of higher orders than the minimal order and whose coefficients are known modulo a prime. To factorize such large order linear differential operators, we make use of the method sketched in section 4. This is done by 'following' the series pertinent to a specific local exponent at a given singular point. Linear combinations of series with different local exponents can be studied as well. Our approach is similar to that proposed by van Hoeij in [15] (a 'local' factorization deduced from formal series analysis around each singularity followed by a 'Hermite–Padé approximation' to obtain the 'global' factorization). The main difference is[11] that in our case the operators to be factorized are defined over a field of positive characteristic[12].

[9] Of course the minimal order operator right divides all these L_{QD}.

[10] The sizes (order and degree) of minimal order versus optimal ODEs are very well understood in some particular cases. For instance, the minimal order ODE (also called 'differential resolvent') satisfied by an *algebraic function* has coefficients whose degree is cubic in the degree of the function, while there exists a linear differential equation of order linear in the degree whose coefficients are only of *quadratic* degree [14]. To our knowledge, analogous estimates do not exist yet for the (more general) case of linear ODEs satisfied by integrals of algebraic functions, such as $\tilde{\chi}^{(n)}$.

[11] Note that the DFactor routine from the DEtools Maple package, corresponding to an implementation of these *local-to-global* ideas [15], fails to factor L_6 of $\tilde{\chi}^{(3)}$. The method we display in section 4 actually succeeds in finding this factorization.

[12] Note that in principle one could also resort to algorithms specially dedicated to factoring linear differential operators modulo a prime p [16]. However, at present these algorithms are far from being efficient enough to handle factorizations modulo primes as big as $p_r = 2^{15} - 19$.

Actually, the modular nature of our calculation is of great help in this since, with the coefficients being known modulo a prime, the coefficients in the linear combination of solutions with given local exponents can take only a finite number of integer values, so that 'guessing' the correct combination can be done by exhaustive search. For each series used as a candidate to 'break' the linear differential operator under consideration we compute three (or more) linear ODEs and from the ODE formula (26) *we infer the minimal order*.

Another point that we address in this paper is the 'complexity' of the linear differential operators corresponding to $\tilde{\chi}^{(n)}$ seen through the various factors occurring in the factorization. One notes that for $\tilde{\chi}^{(4)}$, the factors are either of order one, or are symmetric powers of the linear differential operator L_E. In contrast, the linear differential operator for $\tilde{\chi}^{(3)}$ contains a factor Z_2 of order two which is not equivalent to the linear differential operator L_E. Recently it has been shown [6] that the solution of the linear ODE corresponding to Z_2 is a hypergeometric function (up to a Hauptmodul pull-back) corresponding to a *weight-1 modular form* (see [17]). We think it is important to examine whether the linear ODE for $\tilde{\chi}^{(5)}$ contains other factors, besides various factors equivalent to symmetric powers of L_E, such as the factor Z_2 occurring for $\tilde{\chi}^{(3)}$, that may have a *modular form interpretation*.

Finally we wish to emphasize that the linear differential operator L_{33} is globally nilpotent[13] since it corresponds to a linear ODE that annihilates an integral of an algebraic integrand (3) (it is *'derived from geometry'*, DFG, see [6] and references therein). While this paper is not concerned with global nilpotence as such it must be emphasized that the nilpotent condition places very severe restrictions on a linear differential operator, and in particular, *provides a proper framework*[14] *for the existence of basis of series solutions modulo primes* for the ODEs (see theorem 4 in [21]). Furthermore, we use two important consequences of the global nilpotence of L_{33}. First, globally nilpotent operators are necessarily Fuchsian and permit only rational local exponents. Second, if L is globally nilpotent so is any factor of L.

3. Working with non-minimal order linear differential operators

Once one introduces the particular linear combination (14) of $\tilde{\chi}^{(1)}$ and $\tilde{\chi}^{(3)}$ and (modulo the prime p_r) of $\tilde{\chi}^{(5)}$, it is sufficient to focus on the resulting series and its linear ODE (with the operator L_{29}). From the linear ODE for $\tilde{\chi}^{(5)}$ it is straightforward to obtain the (linear) recursion for the series coefficients and using the combination (14) calculate as many terms[15] as required for $\Phi^{(5)}$. It is thus not difficult to obtain minimal order ODEs requiring fewer terms than p_r. We continue however, as in [7], to work with non-minimal order ODEs for which fewer series terms are needed than for the minimal order ODE. In particular, we make use of the ODE formula (see appendix B) to infer the order and degree of the minimal order ODE. This formula also enables us to *control the minimum number of series terms* necessary to find a Fuchsian linear differential equation (of a given order Q and degree D) which annihilates the series. In the following, when we say that a linear ODE of order q has been obtained, we

[13] The mathematical concept of global nilpotence while quite formal is nevertheless easy to explain. First, if p is a fixed prime number, then the differential operator L is said to have nilpotent p-curvature if and only if modulo p, it right divides the pure power $D_w^{p\cdot \mathrm{ord}(L)}$ of the derivation ($D_w = \mathrm{d}/\mathrm{d}w$). Second, L is called 'globally nilpotent' if it has nilpotent p-curvature modulo almost all prime numbers p (all primes except a finite number of prime).

[14] The Cauchy-Peano theorem [20], which guarantees the existence of series solutions in the classical study of ODEs, *does not apply to linear differential equations in positive characteristic*! This means that in general, a linear differential equation considered modulo a prime number p *does not admit a basis of power series solutions modulo* p*, even at an ordinary point*.

[15] Note that the number of coefficients of all the series used in our calculations does not exceed the value of the prime p_r.

mean that we have obtained at least three non-minimal order ODEs and that the minimal order q has been inferred from the ODE formula.

From the series (14) one can obtain many non-minimal order linear ODEs and the resulting ODE formula for L_{29} reads

$$N = 68Q + 29D - 706 = (Q + 1)(D + 1) - f. \tag{27}$$

Our understanding of the ODE formula (see appendix B and in particular (B.4)) enables us to find $D_{\text{app}} = 1169$ as the degree of the apparent polynomial for the minimal order operator L_{29} without actually producing this minimal order operator.

In [7] we found that there is a simple rational solution of the linear ODE corresponding to L_{30} (and now also L_{29}) which is the square of $\tilde{\chi}^{(1)}$,

$$(\tilde{\chi}^{(1)})^2 = \frac{w^2}{(1 - 4w)^2}, \tag{28}$$

whose corresponding linear differential operator we denote L_1^s.

In this paper we use linear ODEs that are *not* of minimal order to represent the minimal order linear ODE annihilating a given series. We also compute the local exponents at various singular points of the non-minimal linear ODE and consider them as local exponents of the minimal order linear ODE. A remark is in order here. The local exponents at $w = 0$ of the linear ODE corresponding to L_{29} are[16]

$$\rho = 1^5, 2^4, 3^4, 4^3, 5^3, 6^3, 7^2, 8, 9^2, 10, 12. \tag{29}$$

In our computation, the non-minimal linear ODE that represents L_{29} is of order 51. One should then really obtain 51 local exponents. It so happens that the 22 'spurious' exponents appear in the indicial equation as roots of polynomials in ρ of degree two and higher. These exponents are *not rational* (indicial polynomials modulo a prime of degree higher than one and irreducible) and therefore cannot be local exponents for L_{29}, which is globally nilpotent and, hence, has only rational exponents [6]. Had the indicial equation of the non-minimal linear ODE given more than 29 rational exponents then we would have had to produce other non-minimal linear ODEs and obtain the local exponents of L_{29} as those common to all the non-minimal ODEs.

4. Factorization of differential operators versus local exponents

Let us start by giving a brief overview of our factorization procedure. For a linear ODE of order q (which may be known exactly or modulo a prime) we compute the local exponents at a given singular point $w = w_s$ (such as the origin). We create a series $S_p(w)$ starting with the highest integer exponent n_p (we seek mainly to utilize only those solutions analytic at the origin). This series can be obtained to arbitrary length (though shorter than the prime in use) in linear time since we have the linear ODE and, hence, the recursion for the series coefficients. We then check to see whether the particular solution $S_p(w)$ is the solution of a linear ODE of order less than q. If so, the procedure is repeated for each new factor in turn. If not, we generate the series $S_{p-1}(w)$ starting at the second highest exponent n_{p-1}. The series $S_{p-1}(w)$ contains, via a free parameter, a linear combination of the solution $S_p(w)$. We then let the free coefficient of the linear combination vary over the whole (finite) interval $[1, p_r]$, given by the prime p_r we are using, until a linear ODE of order less than q (if such an ODE exists) is found. And then the procedure is repeated.

[16] Throughout the paper the multiplicity of an exponent is denoted by a power: $2, 2, 2, 2 \rightarrow 2^4$.

For a linear ODE of order q, let L_q denote the corresponding differential operator. Consider a singular point $w = w_s$ (for instance $w_s = 0$) and assume the local exponents at this point are

$$\rho_1^{m_1}, \rho_2^{m_2}, \ldots, \rho_p^{m_p}, \qquad \sum_{j=1}^{p} m_j = q, \tag{30}$$

where m_j is the multiplicity of the exponent ρ_j. In our cases the exponents are either integers or rational numbers. Here we utilize only solutions, which are analytic at the singular point w_s. So in what follows we consider only integer exponents and we denote these as n_p. We can then plug the series

$$S_p(w) = w^{n_p} + \sum_{k \geqslant n_p+1} a_k w^k, \tag{31}$$

into the linear ODE. Demanding $L_q(S_p(w)) = 0$ will fix all coefficients a_k. By producing enough terms we can find the linear ODE for the particular series solution $S_p(w)$, which is by construction a solution of L_q. The resulting ODE will either have order q or order $q_1 < q$. In the first case this could mean that L_q is irreducible, or L_q does factorize but the factor 'responsible' for annihilating the solution $S_p(w)$ is a left-most factor. In the second case we have the factorization

$$L_q = L_{q-q_1} \cdot L_{q_1}. \tag{32}$$

To summarize, the series corresponding to the highest local exponent leads to either the full ODE or to a 'breaking' of the original ODE.

If the series $S_p(w)$ (corresponding to the highest local exponent) reproduces the full linear ODE we turn to the second highest exponent n_{p-1}. In this case, a series starting as $w^{n_{p-1}} + \cdots$, plugged into the original linear ODE, will yield the expansion

$$S_{p-1}(w) = w^{n_{p-1}} + \sum_{k \geqslant n_{p-1}+1}^{n_p - 1} a_k w^k + a_{n_p} w^{n_p} + \sum_{k \geqslant n_p+1} c_k w^k, \tag{33}$$

where all a_k up to (but not including) a_{n_p} are fixed and c_k's depend linearly on the free coefficient a_{n_p}, i.e. $S_{p-1}(w)$ is a one-parameter solution. The series $S_{p-1}(w)$ is a sum of a series starting as $w^{n_{p-1}} + \cdots$ and the series[17] $a_{n_p} S_p(w)$. For generic values of the coefficient a_{n_p} the series $S_{p-1}(w)$ will give rise to the full linear ODE. But for some values of the coefficients a_{n_p}, the series $S_{p-1}(w)$ may be the solution of a linear ODE of order less than q. This is what leads to the factorization of L_q. Intuitively we may hope that such a procedure can work for the following reason. If the original operator has many smaller factors this would indicate that there is a basis of solutions much simpler than those requiring the full ODE. We do not know this basis but by taking a linear combination of two formal 'full' solutions (which obviously are linear combinations of the basis solutions) it is possible that we can find values of the combination coefficients such that the resulting series is a solution of a simpler ODE (for these special values some of the basis solutions from the two formal solutions cancel each other).

Similarly, the series solution of L_q that starts at the third highest exponent n_{p-2} will be a two parameters solution (for simplicity we assume that the exponents differ by one)

$$S_{p-2}(w) = w^{n_{p-2}} + a_{n_{p-1}} w^{n_{p-1}} + a_{n_p} w^{n_p} + \sum_{k \geqslant n_p+1} c_k w^k, \tag{34}$$

where the c_k depend linearly on both the free coefficients $a_{n_{p-1}}$ and a_{n_p}.

[17] Alternatively we can view this procedure as looking at a linear combination of the two formal series solutions starting as $w^{n_{p-1}} + \cdots$ and $w^{n_p} + \cdots$ respectively.

To demonstrate how the procedure works in practice we consider the seventh-order linear ODE for $\tilde{\chi}^{(3)}$ [10, 12] (denoted L_7) . At the singular point $w = 0$, the local exponents are

$$1^3, 2^2, 3, 9. \tag{35}$$

Acting with the linear ODE for $\tilde{\chi}^{(3)}$ on the series that starts as $w^9 + \cdots$, (i.e. with the highest exponent)

$$S_9(w) = w^9 + \sum_{k \geqslant 10} a_k w^k, \tag{36}$$

fixes all the coefficients. We thus obtain the expansion at $w = 0$ of $\tilde{\chi}^{(3)}/8$, leading to the full linear ODE. Of course, this is not surprising, since the series for $\tilde{\chi}^{(3)}$ used to 'generate' the linear differential operator L_7 starts as w^9 as per (8), so the unique series $S_9(w)$ must be proportional to $\tilde{\chi}^{(3)}$. The series $S_9(w)$ cannot be used to 'break' L_7, since this is the minimal order operator annihilating $\tilde{\chi}^{(3)}$.

Consider next a series that starts as $w^3 + \cdots$, i.e. with the second highest exponent

$$S(w) = w^3 + \sum_{k \geqslant 4} a_k w^k. \tag{37}$$

We insert this series into the exact ODE for $\tilde{\chi}^{(3)}$ and then we solve (term by term) the equations arising from $L_7\,(S_3(w)) = 0$. Doing this we find that the coefficients $a_4, a_5, \ldots, a_8$ are fixed while the coefficient a_9 remains undetermined and hence enters the series as a free parameter. The remaining coefficients are all given in terms of a_9

$$\begin{aligned} S(w) = w^3 + 3w^4 + 22w^5 + 74w^6 + 417w^7 + 1465w^8 + a_9 w^9 \\ + 26839w^{10} + (36a_9 - 139067) \cdot w^{11} + (4a_9 + 443325) \cdot w^{12} + \cdots. \end{aligned} \tag{38}$$

The terms in $S(w)$ in front of the free coefficient a_9 are the coefficients of the series $S_9(w)$. We define $S_3(w)$ to be the series obtained from $S(w)$ by setting $a_9 = 0$. In order to break the operator L_7 we look at linear combinations $S_\alpha(w) = S_3(w) + \alpha S_9(w)$. For generic values of α the series $S_\alpha(w)$ is annihilated only by the full ODE of order seven. However, it is possible that for special values of α the series $S_\alpha(w)$ is the solution of a linear ODE of order less than seven.

We do not know if the 'splitting' values of α can be obtained except by a 'guessing' procedure. The use of modular calculations is very useful in the search for the special splitting values. The series $S_9(w)$ and $S_3(w)$ can be obtained modulo any prime p_r and in the modular calculations α can take its value only in the finite range $[1, p_r]$. If a rational splitting value of α exists it can be found by looking for an underlying ODE of order less than 7 annihilating the series $S_\alpha(w)$. In the search we use the optimal ODE, which is of order 10 and degree 19 with $N = 213$. We used the prime $p_r = 32749$ in our search. For each value of $\alpha \in [1, 32749]$ we calculated the series modulo p_r and then looked for an annihilating ODE of order 10 and degree 19. For any value of α such an ODE exists and for almost all values $N = 213$. However, for the special values $\alpha = 7463$ and 7467 we have $N = 140$ and 206, respectively. The decrease in N is a sure sign that a simpler ODE annihilates $S_\alpha(w)$. In this particular case we find that the ODE for $\alpha = 7463$ is of order four while for $\alpha = 7467$ the ODE is of order six.

In the case $\alpha = 7463$ the linear differential operator is $X_1 \cdot Z_2 \cdot N_1 = B_2 \cdot O_1 \cdot N_1 = B_2 \cdot T_1 \cdot L_1$, while in the case $\alpha = 7467$ the linear differential operator is $Y_3 \cdot Z_2 \cdot N_1$. These linear differential operators are factors of L_7 that were already found in [13] (the indices indicate the orders of the corresponding linear differential operators)

$$L_7 = M_1 \cdot Y_3 \cdot Z_2 \cdot N_1 = B_3 \cdot X_1 \cdot Z_2 \cdot N_1$$
$$= B_3 \cdot B_2 \cdot O_1 \cdot N_1 = B_3 \cdot B_2 \cdot T_1 \cdot L_1. \tag{39}$$

Remark. We note that the method is not special to formal series. *A fortiori*, it applies to linear combinations of solutions *suspected* of being parts of a direct sum. For instance, removing the series $\alpha\tilde{\chi}^{(1)}$ from the series (modulo a prime) $\tilde{\chi}^{(5)} - \frac{1}{2}\tilde{\chi}^{(3)}$ (see (13)), and, letting α vary in the interval $[1, p_r]$ (recall that α which is a rational number appears as an integer modulo a prime), will give (for one value of α) a linear ODE of order 29 if the linear differential operator L_1 for $\tilde{\chi}^{(1)}$ is in a direct sum in L_{30}, the linear differential operator for $\tilde{\chi}^{(5)} - \frac{1}{2}\tilde{\chi}^{(3)}$. If L_1 had not been part of a direct sum the outcome would have been an order 30 linear ODE for *all* values of α.

5. Factorization modulo a prime of the linear differential operator L_{29}

We turn now to the factorization of L_{29} for which we know that $L_1^s \oplus (Z_2 \cdot N_1)$ is a factor. We focus solely on the analytical solutions at $w = 0$ and we first produce the unique series that starts as $S_{12}(w) = w^{12} + \cdots$, where the coefficients in $S_{12}(w)$ are given by $L_{29}(S_{12}(w)) = 0$. We found that $S_{12}(w)$ is the solution of an order nine linear ODE (with linear differential operator L_9) with ODE formula

$$N = 18Q + 9D - 73 = (Q + 1)(D + 1) - f. \tag{40}$$

We know that both $(\tilde{\chi}^{(1)})^2$ and the solutions of the linear ODE corresponding to $Z_2 \cdot N_1$ occurring in L_7, should be in L_{29}. By explicit checking we found that only L_1^s and N_1 are factors of L_9. We can then add the solutions of $Z_2 \cdot N_1$ to the solutions of L_9 to produce an 11th-order linear ODE (denoted L_{11}). At the operator level this is done by a direct-sum construction $L_{11} = L_9 \oplus (Z_2 \cdot N_1)$. At the series level used in the ODE search programs, it can be done by creating a 'generic' solution of L_9 (generic means a series that gives the full ODE) and then forming a linear combination with a generic solution of $Z_2 \cdot N_1$ to produce a series which is a generic solution of L_{11}. The resulting linear differential operator L_{11} has the ODE formula

$$N = 24Q + 11D - 111 = (Q + 1)(D + 1) - f. \tag{41}$$

We have thus shown the following factorization of L_{29}:

$$L_{29} = L_{18} \cdot L_{11}. \tag{42}$$

Before proceeding we wish to clarify the meaning of the ODE corresponding to the left factors. Having obtained the linear differential operators L_{29} and L_{11}, a right division should give the linear differential operator L_{18}. One should bear in mind that the order of these operators is large and our representation of them are not of minimal order. In the computation, the linear differential operators representing L_{29}, L_{18} and L_{11} are of orders 51, 32 and 17, respectively. Our representation of the factorization (42) reads in fact

$$O_{22} \cdot L_{29} = (O_{14} \cdot \tilde{L}_{18}) \cdot (O_6 \cdot L_{11}), \tag{43}$$

where the equality stands for 'both sides acting on $S(w)$ give zero'. Since the series solution $S(w)$ *demands* an order 29 linear ODE, and the order of the extra operator O_6 is arbitrary, there are intertwinners leading to

$$O_{22} \cdot L_{29} = O_{14} \cdot \tilde{O}_6 \cdot (L_{18} \cdot L_{11}). \tag{44}$$

With the relation $\tilde{L}_{18} \cdot O_6 = \tilde{O}_6 \cdot L_{18}$, the linear differential operators $\tilde{L}_{18}$ and L_{18} are equivalent and have the same factorization structure.

Next we take the series $S(w) = \tilde{\chi}^{(5)} - \tilde{\chi}^{(3)}/2 + \tilde{\chi}^{(1)}/120$ and plug it into the linear ODE for L_{11} to produce a new series whose linear ODE corresponds to the linear differential operator L_{18}. This linear ODE has the formula

$$N = 44Q + 18D + 873 = (Q + 1)(D + 1) - f. \tag{45}$$

To proceed further with the factorization, we compute the local exponents at $w = 0$ for the linear ODE corresponding to L_{18}. These are

$$\rho = 1^3, 2, 3^2, 4, 5^2, 6^3, 7^2, 8, 9^2, 10. \tag{46}$$

For the linear differential operator L_{18} we look at the solution that starts as $S_{10}(w) = w^{10} + \cdots$. Unfortunately, this gives linear ODEs with the same ODE formula as L_{18}, that is the series reproduces the complete linear ODE represented by L_{18}. This means that the factor responsible for the annihilation of $S_{10}(w)$ occurs at the left of L_{18}.

The second highest exponent is $\rho = 9$. When L_{18} is applied to a generic series $S_9(w) = w^9 + \cdots$ we obtain a series that depends on the coefficient in front of w^{10}. This one-parameter series starts as (modulo the prime p_r)

$$w^9 + a_{10}w^{10} + (15\,419a_{10} + 10040) \cdot w^{11} + \cdots. \tag{47}$$

The series, collected in a_{10}, enables us to reconstruct the full linear differential operator L_{18}, but it is possible that the above combination may give a linear operator of *smaller order* for particular values of a_{10}, which have to be found by experimentation. It is here that the modular calculations allow us to find a definitive answer. Modulo the prime p_r, the coefficient a_{10} spans a finite set of integer values $[1, p_r]$. The determination of the coefficient a_{10} is thus feasible in a finite computational time by exhaustive search.

With the value $a_{10} = 12\,999$ we found that the series S_9 gives a linear ODE of *smaller order* than L_{18} with the ODE formula

$$N = 36Q + 13D + 715 = (Q + 1)(D + 1) - f. \tag{48}$$

The local exponents at $w = 0$ for this linear ODE of order 13 (that we denote L_{13}) are

$$\rho = 1^2, 2, 3, 4, 5^2, 6^2, 7^2, 8, 9. \tag{49}$$

The highest exponent is indeed nine, of which the associated solution gave us L_{13}.

Let us be more explicit on the meaning of the combination $w^9 + 12\,999w^{10} + \cdots$ that has given rise to the 13th-order linear ODE. By 'following' the series $w^{10} + \cdots$ we obtained the full linear ODE. The factor that annihilates this series is thus to the left in the factorization of L_{18}. We find that this series comes with a $\log(w)^4$ behaviour (see below). By the combination $w^9 + 12\,999w^{10} + \cdots$, we are looking for the particular value of a_{10} that *removes* this logarithmic solution from the linear ODE corresponding to L_{18}. Thus the linear ODE for L_{13} no longer has a solution behaving as $w^{10}\log(w)^4$.

Completing from L_{13} to L_{18} we obtain a fifth-order linear ODE (called L_5) with the ODE formula

$$N = 8Q + 5D + 912 = (Q + 1)(D + 1) - f. \tag{50}$$

We thus have the factorization

$$L_{29} = L_5 \cdot L_{13} \cdot L_{11}. \tag{51}$$

The fifth-order linear differential operator L_5 is that whose existence we conjectured previously and which we believe should annihilate a homogeneous polynomial of the complete elliptic integrals E and K of (homogeneous) degree four.

The local exponents at the origin of the linear ODE corresponding to L_5 are

$$w = 0, \qquad \rho = 1, 3, 6, 9, 10. \tag{52}$$

Plugging a generic series $\sum c_n w^n$ into the linear ODE fixes all the coefficients (including c_1, c_3, c_6, c_9) with the exception of the coefficient c_{10}. The 'survival' of a single coefficient is a particular feature of an irreducible factor with one non-logarithmic solution. The exponents at the other singularities (apart from $w = \infty$) are

$$w = 1/4, \qquad \rho = -29^2, -28, -23, 0,$$
$$w = -1/4, \qquad \rho = -35^2, -33, -31, 0.$$

This suggests that one should plug the following ansatz into the linear ODE:

$$\frac{1}{(1-4w)^{29}(1+4w)^{35}} \cdot \sum_{i=0}^{4} P_{4-i,i} \cdot K^{4-i} E^i, \tag{53}$$

where K and E denote the complete elliptic integrals

$$K = {}_2F_1([1/2, 1/2], [1], 16w^2), \qquad E = {}_2F_1([1/2, -1/2], [1], 16w^2).$$

Collecting terms of the form $K^{4-i}E^i$ we determine the polynomials $P_{4-i,i}$ whose degrees are increased steadily until we obtain a solution[18]. With degree around 200 the following solution was found:

$$\frac{w}{(1-4w)^{29}(1+4w)^{35}} \cdot ((1-16w^2)^3 P_{4,0} \cdot K^4 + (1-16w^2)^2 P_{3,1} \cdot K^3 E$$
$$+ (1-16w^2) P_{2,2} \cdot K^2 E^2 + P_{1,3} \cdot K E^3 + P_{0,4} \cdot E^4),$$

where $P_{4-i,i}$ are polynomials in w with coefficients known modulo the prime p_r and of degree respectively 200, 202, 204, 204 and 204. The expressions for the polynomials $P_{4-i,i}$ can be found in [22]. As conjectured the linear differential operator L_5 *is thus equivalent to the symmetric fourth power of* L_E.

5.1. The linear differential operator L_{11} has six factors

As shown above the linear differential operator L_{29} factorizes into three factors of order 11, 13 and 5. We have just shown that the fifth-order linear differential operator is irreducible. Next we consider the linear differential operator L_{11}.

We know that the fourth-order linear differential operator $L_1^s \oplus (Z_2 \cdot N_1)$ is a right-most factor of L_{11}, so we obtain

$$L_{11} = N_7 \cdot \left(L_1^s \oplus (Z_2 \cdot N_1)\right). \tag{54}$$

The ODE formula for the seventh-order linear differential operator N_7 reads

$$N = 15Q + 7D + 89 = (Q+1)(D+1) - f. \tag{55}$$

At $w = 0$, the local exponents (for N_7) are

$$\rho = 2^2, 3, 4^2, 5, 12. \tag{56}$$

Plugging the series $w^5 + \sum_{k\geqslant 6} a_k \cdot w^k$ into the linear ODE for N_7 fixes all the coefficients except a_{12} corresponding to a solution with the highest local exponent. Letting the combination coefficient a_{12} vary in the finite range $[1, p_r]$, we found for the value $a_{12} = 22\,292$ a linear ODE of order less than seven, with the ODE formula

$$N = 13Q + 5D + 79 = (Q+1)(D+1) - f, \tag{57}$$

[18] Once the solution has been obtained a check to any order can be carried out. Typically a good check amounts to plugging polynomials of degree 300 into (53).

and with exponents at the origin

$$\rho = 2^2, 3, 4, 5. \tag{58}$$

For this linear ODE we consider the one-parameter series that starts with w^4 and which contains the coefficient a_5 as a 'free' parameter. By letting the coefficient a_5 vary in the finite range $[1, p_r]$, we found that for $a_5 = 29\,103$, the linear ODE of order five breaks into two linear differential operators of order three and two, $O_3 \cdot O_2$, respectively.

In conclusion we have decomposed the differential operator L_{11} of order 11 into six irreducible factors

$$L_{11} = \tilde{O}_2 \cdot O_3 \cdot O_2 \cdot \left(L_1^s \oplus (Z_2 \cdot N_1)\right), \tag{59}$$

where the indices denote the order of the corresponding linear differential operators.

5.2. The linear differential operator L_{11} in exact arithmetic

Nickel has obtained [32] some linear differential operators that right divide L_{30} and, especially, the linear differential operators (that we call) $U_2 \cdot N_1$ and $F_3 \cdot F_2 \cdot L_1^s$. Checking these operators against the factorization (59), we found that L_{11} has the direct-sum decomposition

$$L_{11} = (Z_2 \cdot N_1) \oplus V_2 \oplus \left(F_3 \cdot F_2 \cdot L_1^s\right), \tag{60}$$

where V_2 is equivalent to U_2 (or to $\tilde{O}_2$) and the product $F_3 \cdot F_2$ is equivalent to the product $O_3 \cdot O_2$ in (59).

Furthermore, using some tricks in the modular calculations supplemented with constraints on the apparent polynomials ([23], appendix A in [7]), Nickel succeeded in finding the considered linear differential operators exactly. The linear differential operators V_2, F_2 and F_3 are given[19] in appendix C.

The procedure for obtaining a rational number from its image modulo a prime is known as '*rational reconstruction*' and has many applications (for details see, e.g. [24–26]). Consider a rational number n/d which has the residue u modulo the prime m. Given u and m, a rational reconstruction algorithm tries to recover the rational n/d under some conditions on the magnitude of the unknowns n and d. The simple version of this condition is

$$2N^2 < m, \qquad N = \max(|n|, d). \tag{61}$$

The algorithm will then output the rational n/d satisfying the above condition, but this rational number may not be the actual one for the problem. If the residue is known for several primes m_i, it is the *Chinese remainder* u which is considered and m is the product of the primes m_i.

In any case, knowledge of the order of magnitude of n and d is important. For instance, the exact linear differential operator F_2 can be recovered using the results *for three primes*[20]. In our modular calculations the residues are coefficients of polynomials occurring in Fuchsian linear ODEs. Besides the order of magnitude, which can be guessed, the linear ODE, once reconstructed, should satisfy certain properties. The indicial equation should give the correct local exponents, which have been obtained either from a linear ODE modulo a prime or from a diff-Padé analysis [7]. For the apparent singularities the conditions on the apparent polynomial should be verified.

Consider the linear differential operator F_2 (see appendix C)

$$F_2 = \mathcal{P}_2(w) P_{\mathrm{app}}(w) \cdot \mathrm{D}_w^2 + \mathcal{P}_1(w) P_1(w) \cdot \mathrm{D}_w + P_0(w). \tag{62}$$

[19] We are grateful to B G Nickel for these results.

[20] We have actually obtained the linear differential operator $L_{29} = L_{18} \cdot L_{11}$ for four primes $2^{15} - 19$, $2^{15} - 49$, $2^{15} - 51$ and $2^{15} - 55$.

The singularities are known and are roots of the polynomial $\mathcal{P}_2(w)$ (the polynomial $\mathcal{P}_1(w)$ contains a subset of the singularities). We note that $P_{\rm app}(w)$ (the apparent polynomial) and $P_1(w)$ can be reconstructed with two primes while $P_0(w)$ demands three primes[21].

However, when $P_{\rm app}(w)$ and $P_1(w)$ have been found, the matching of the known local exponents will fix some coefficients in $P_0(w)$, but, more importantly, we get an idea about the order of magnitude of the common denominator in the various rational coefficients in $P_0(w)$. This magnitude was found to be 2^{16} and, with this scaling, the still unreconstructed coefficients in $P_0(w)$ can be reconstructed using two primes and then checked against the conditions for the apparent singularities. Our iterative procedure for reconstructing the exact polynomials thus amounts to first reconstructing the polynomials $P_{\rm app}(w)$ and $P_1(w)$ (with two primes) in order to obtain $P_0(w)$ with two primes instead of three. More details can be found in appendix D, which deals with a rational reconstruction experiment on the apparent polynomial of F_3. One should note that, to rationally reconstruct a linear ODE, it must be obtained with as many primes as necessary. When the results are time consuming or hard to obtain, knowledge of the underlying problem may help one to guess the scaling factor which forces the condition (61).

Note that when the number of primes is not sufficient some reconstructed coefficients will, obviously, be in error. A strong check can finally be done on these linear differential operators that should convince one of their correctness. The linear differential operators F_2 and F_3, that we are looking for in exact arithmetic, are factors of the linear differential operator L_{33}. They are *necessarily globally nilpotent* [6] since L_{33} is. The linear differential operator L_{33} is globally nilpotent since it corresponds to a linear ODE that annihilates an integral of an algebraic integrand (3) (it is *'derived from geometry'*, DFG, see [6] and references therein). We have calculated the p-curvatures of these reconstructed linear differential operators F_2 and F_3, and found that they are, indeed, globally nilpotent.

The global nilpotence of a linear differential operator is a strong and *very special* property that is rigid enough to make us *totally confident* that the polynomials occurring in the linear differential operators F_2 and F_3 have been reconstructed correctly.

As for the solutions of the factors occurring in L_{11}, the simple V_2 is equivalent to the linear differential operator L_E and its linear ODE annihilates

$$\frac{w^2}{1-4w}\cdot\left(K-\frac{2E}{1-16w^2}\right). \qquad (63)$$

From (63) it is straightforward to see that V_2 is actually equivalent to the second-order operator for $\tilde{\chi}^{(2)}$ (denoted N_0 in [11]). Note that V_2, equivalent to N_0, does *not* mean that $\tilde{\chi}^{(2)}$ itself is a solution of L_{29}, but rather some linear combination of $\tilde{\chi}^{(2)}$ and its first derivative is. Indeed (63) can be expressed as

$$\frac{(1+4w)(1+8w^2)}{2w}\cdot\frac{{\rm d}\tilde{\chi}^{(2)}}{{\rm d}w}-4\cdot(1+4w)\cdot\tilde{\chi}^{(2)}.$$

The remarkably simple result (63) begs the question about the very nature of V_2. Is the occurrence of $\tilde{\chi}^{(2)}$ (and its first derivative) in $\tilde{\chi}^{(5)}$ a mere coincidence or does it suggest a more general structure. Does an operator equivalent to $\tilde{\chi}^{(4)}$ appear in $\tilde{\chi}^{(7)}$ (or $\tilde{\chi}^{(3)}$ in $\tilde{\chi}^{(6)}$)? If we do not have this very strong result, is it nevertheless the case, that some of the individual factors occurring in say the factorizations of L_7 appears in $\tilde{\chi}^{(6)}$? Expressed more generally is it the case that operators equivalent to factors from $\tilde{\chi}^{(m)}$ appear in the factorization of $\tilde{\chi}^{(n)}$ (with $m \leqslant n$ and n and m of different parity)? Similar questions can be asked with regard to the occurrence of the rational solution of L_1^s that we have written as $(\tilde{\chi}^{(1)})^2$ in (28). We have already conjectured in (23) that for $\tilde{\chi}^{(n)}$'s we have direct-sum structures corresponding

[21] We have in each case one or more additional results modulo a prime for checking our calculations.

to linear ODE's of smaller order for selected linear combinations involving $\tilde{\chi}^{(n-2k)}$. Could it be that polynomial (i.e. nonlinear) combinations of $\tilde{\chi}^{(m)}$'s $(m < n)$ can be used to further deplete $\tilde{\chi}^{(n)}$ or at least appear as factors?

As emphasised in [6], we have a strong belief (but no proof) that all the linear differential operators occurring as factors of the linear differential operators for $\tilde{\chi}^{(n)}$ s, have to *be related to the theory of elliptic curves* (complete elliptic integrals E and K, algebraic modular functions expressed as algebraic hypergeometric functions, *modular forms of weight-1*, etc). The calculation of the p-curvature of F_2 and F_3 excludes linear differential operators associated with algebraic functions. The simple occurrence in (63) of E and K is in contrast to the linear differential operators F_2 and F_3, which do not seem to be equivalent to a symmetric power of L_E. We must explore whether (similarly to what we found [6] for Z_2 and the linear differential operator occurring in the analysis[22] of $\Phi_H^{(3)}$) the linear differential operators F_2 (respectively F_3) correspond to modular forms of weight-1 (or higher weights), or ${}_2F_1$ (respectively ${}_3F_2$) hypergeometric functions with a Hauptmodul pull-back (up to multiplication by some n th root of a rational function).

To see if a solution of the second-order operator F_2 is a ${}_2F_1$ hypergeometric function with a Hauptmodul pull-back (up to multiplication by the n th root of some rational function) would require one to find not only the Hauptmodul pull-back, but also a change of variable (covering) mapping the large set of singularities in F_2 onto three singularities $(0, 1, \infty)$, and find, besides, the rational function occurring in the multiplicative factor in front of the hypergeometric function ${}_2F_1$ (which looks like the Hauptmodul [6]).

The occurrence of an involved apparent polynomial is a quite severe obstruction for performing these educated guesses. It is always possible to get rid of the apparent polynomial of a linear differential operator by introducing a higher order, but still Fuchsian, linear differential operator with no apparent singularities (see appendix C). This however is not helpful. What we really need is to find an equivalent linear differential operator with a smaller apparent polynomial or, hopefully, no apparent polynomial.

This is how we achieved [6] such a calculation for the linear differential operator Z_2. We were able to find a second-order operator (occurring as a factor in $\Phi_H^{(3)}$), which is simpler than Z_2 because its apparent polynomial is just $1 - 2w$

$$Z_2 \cdot M_1 = \tilde{M}_1 \cdot \tilde{Z}_2, \tag{64}$$

where M_1 and $\tilde{M}_1$ are two first-order linear differential intertwinners. Up to the change of variable (covering)

$$x = \frac{72w}{(1-w)(1-4w)}, \tag{65}$$

wrapping the seven singularities of Z_2 onto the three singularities of the hypergeometric function, we were able to find a *modular form of weight-1* solution of the equivalent second-order operator $\tilde{Z}_2$. This was a consequence of its very simple apparent polynomial. At the present moment, we have not been able to replace F_2 by an equivalent second-order operator with a simpler apparent polynomial. The situation is even more involved for F_3 (see appendix C). The modular form interpretation of F_2 and F_3 remains to be done and is clearly a worthy challenge.

[22] The functions $\Phi_H^{(n)}$ are simplified Ising type integrals [9] obtained from the $\tilde{\chi}^{(n)}$ integral representation (3) by setting the Fermionic factor $G^{(n)} = 1$.

5.3. The linear differential operator L_{13}

We continue our procedure of factorization for the 13th-order operator L_{13} occurring as a factor in L_{29}. Recalling the local exponents at $w = 0$

$$\rho = 1^2, 2, 3, 4, 5^2, 6^2, 7^2, 8, 9 \qquad (66)$$

we know that the series corresponding to the highest exponent enables one to reconstruct the full linear ODE. The next series to consider is thus the one-parameter series $w^8 + a_9 w^9 + \cdots$. For every value of the linear combination coefficient a_9 in the interval $[1, p_r]$, we found that the resulting linear ODE is of order thirteen.

Both series ($w^9 + \cdots$ and $w^8 + a_9 w^9 + \cdots$) are annihilated by a left-most factor in L_{13}. To proceed with the factorization, we would have to consider 'deeper' combinations of series solutions (see section 4). For instance at $w = 0$, we could use the two-parameter (a_8, a_9) solution $w^7 + a_8 w^8 + a_9 w^9 + \cdots$, then the three-parameter (a_7, a_8, a_9) solution $w^6 + a_7 w^7 + a_8 w^8 + a_9 w^9 + \cdots$, and finally the four-parameter solution $w^5 + a_6 w^6 + a_7 w^7 + a_8 w^8 + a_9 w^9 + \cdots$. However, if t_0 is the computational time for a single ODE search, then to check the factorization using a solution with k free parameters requires a computational time of $t_0 p_r^k$. This requires a very long time for the prime $p_r = 32\,749$ taking into account the sizes of the linear ODEs we are dealing with here, and hence we have not pursued this approach beyond the one-parameter case. We could also use, from the outset, the most general five-parameter solution $a_5 w^5 + a_6 w^6 + a_7 w^7 + a_8 w^8 + a_9 w^9 + \cdots$, (only $a_5 = 0$ and $a_5 = 1$ need to be considered) that should give all the factorizations (if any) of L_{13}. However, a check of the factorization using this five-parameter series solution clearly suffers from the prohibitive time requirements mentioned above and it is beyond our current computational resources.

In section 4 we described our method of factorization modulo a prime by focusing on the singularity at the origin. This singular point has no special properties which makes it better suited than other singular points for our factorization scheme. However, to have a clear working scheme, the singular point one chooses to focus on must be sufficiently singular (by which we mean, in this case, have many confluent logarithms) to allow one to extend from the *local scheme* to the *global scheme*. So, we looked at what happens if we use expansions about other singular points.

Considering the ODE corresponding to L_{13} translated to[23] $w = \infty$ one can follow the series of the highest exponent which is -30. This series also demands the full ODE. The one-parameter series corresponding to the second highest exponent $x^{-31} + a_{-30} x^{-30} + \cdots$, (with $x = 1/w$) also gives rise to the full ODE (i.e. the order remains 13) for all values of $a_{-30} \in [1, p_r]$. Similar calculations were performed for the linear ODE translated to[24] $w = 1/4$. Neither the series $x^5 + \cdots$, nor the one-parameter series $x^4 + a_5 x^5 + \cdots$, (with $x = w - 1/4$) gives rise to a linear ODE of order less than thirteen for any value of $a_5 \in [1, p_r]$. The series solutions in front of the higher logarithmic solutions around some other singular points give rise to the full linear ODE of order thirteen.

Instead of considering the series solution $a_5 w^5 + a_6 w^6 + a_7 w^7 + a_8 w^8 + a_9 w^9 + \cdots$, with its prohibitive computational time requirements, we decided to try another procedure that may give us an idea about the number and order of factors occurring in L_{13} (if reducible).

We start by examining how the various formal solutions of L_{13} appear. Consider (near $w = 0$) a general solution with log's such as

$$S(w) = S_n(w) \log(w)^n + S_{n-1}(w) \log(w)^{n-1} + \cdots + S_0(w), \qquad (67)$$

[23] The local exponents at $w = \infty$ are $-40, -39^2, -38, -36^2, -35, -34^2, -33, -32, -31, -30$.

[24] The local exponents at $w = 1/4$ are $-9, -8, -7, -6, -5, -4, -3, 0, 1, 2, 3, 4, 5$.

where the exponent n of the log is generically taken to the maximum allowed value of 12, i.e. the order of L_{13} minus one, and where the $S_n(w)$ are power series expansions $a_0^{(n)} + a_1^{(n)} w + \cdots$. Plugging the solution $S(w)$ into the linear ODE corresponding to L_{13} and solving $L_{13}\,(S(w)) = 0$ term by term, we found that the highest allowed exponent is $n = 3$.

We therefore fix the solution $S(w)$ as

$$S(w) = S_3(w)\log(w)^3 + S_2(w)\log(w)^2 + S_1(w)\log(w) + S_0(w), \qquad (68)$$

and act on it by the linear ODE corresponding to L_{13} and solve term by term (we have to solve only to w^9 since this is the highest local exponent for L_{13} around $w = 0$). The coefficients (up to w^9) in the $S_k(w)$ must be fixed. Among the 40 coefficients 13 will remain as free parameters (equal to the order of the linear ODE). Attached to any of these free coefficients is an independent solution of the linear ODE.

To clarify the scheme of these solutions, from which we shall infer the number of factors, we solve $L_{13}(S(w)) = 0$. This leads to the equation

$$\sum_{k\geqslant 0}\left(\sum_{n=0}^{3} C_k^{(n)}\log(w)^n\right) w^k = 0, \qquad (69)$$

which we solve for each k and n by using the following recipe: the coefficient $C_k^{(n)}$ of the term $w^k\log(w)^n$ will in general be a linear combination of coefficients $a_j^{(m)}$ from $S_m(w)$ with $j \leqslant k$ and $n \leqslant m$. When the coefficient $C_k^{(n)}$ contains coefficients $a_j^{(m)}$ with $m = n$ only, we solve for the coefficient $a_j^{(n)}$ of highest index j. When the coefficient $C_k^{(n)}$ contains coefficients $a_j^{(m)}$ with $m \leqslant n$, we solve for the coefficient $a_j^{(m)}$ of lowest index m and highest index j. This is because we know, from all our computations on Ising type ODEs, that when a solution such as $S(w)\log(w)^{(n)} + \cdots$ occurs, $S(w)\log(w)^{(n-1)} + \cdots$, (with the same $S(w)$) is also a solution.

We introduce the notation $[w^p]$ to mean that the series begins as $w^p(\text{const.} + \cdots)$. The results of the computation are the following. Four solutions can be written as

$$\begin{aligned} &[w^7]\log(w)^3 + [w^5]\log(w)^2 + [w]\log(w) + [w],\\ &[w^7]\log(w)^2 + [w^5]\log(w) + [w],\\ &[w^7]\log(w) + [w], \qquad \text{and} \qquad [w^7]. \end{aligned} \qquad (70)$$

Four other solutions can be written as

$$\begin{aligned} &[w^6]\log(w)^3 + [w^5]\log(w)^2 + [w]\log(w) + [w],\\ &[w^6]\log(w)^2 + [w^5]\log(w) + [w],\\ &[w^6]\log(w) + [w], \qquad \text{and} \qquad [w^6]. \end{aligned} \qquad (71)$$

There are also two sets of two solutions each that read

$$[w^9]\log(w) + [w], \qquad \text{and} \qquad [w^9], \qquad (72)$$

and

$$[w^8]\log(w) + [w], \qquad \text{and} \qquad [w^8]. \qquad (73)$$

Finally there is a non-logarithmic solution starting as $w^5 + \cdots$.

In view of this scheme, one may conclude, in analogy with all the Ising calculations we have performed and where hypergeometric functions occur, that the factors occurring in L_{13} are of order 4, 4, 2, 2 and 1. At the point $w = \infty$, one obtains the same structure of solutions leading to the same scheme, that is factors of order 4, 4, 2, 2 and 1. However, at the singular point $w = 1/4$ the structure changes slightly. The solutions, grouped as done above for the point $w = 0$, lead to a scheme of six factors with orders 4, 2, 2, 2, 2 and 1.

To reconcile the three[25] schemes (around $w = 0$, $w = 1/4$ and $w = \infty$), the linear differential operator L_{13} may have either four factors of orders 4, 4, 4 and 1 or five factors of orders 4, 4, 2, 2 and 1.

Around the three singular points, the schemes allow for an order one differential operator whose corresponding series starts as $w^5 + \cdots$. It happens that this first-order differential operator (call it $\tilde{L}_1$) occurs as a right-most factor of $L_{13} = L_{12} \cdot \tilde{L}_1$.

We have found that the solution of the linear ODE corresponding to $\tilde{L}_1$ is a simple polynomial of degree 34, which reads (modulo the prime p_r)[26]

$$\begin{aligned} P(w) = w^5 &+ 30\,849w^6 + 4080w^7 + 11\,244w^8 + 26\,721w^9 \\ &+ 29\,301w^{10} + 23\,070w^{11} + 30\,185w^{12} + 26\,217w^{13} + 10\,853w^{14} \\ &+ 25\,659w^{15} + 4536w^{16} + 31\,400w^{17} + 22\,061w^{18} + 31\,481w^{19} \\ &+ 3767w^{20} + 6508w^{21} + 10\,160w^{22} + 31\,426w^{23} + 29\,441w^{24} \\ &+ 17\,755w^{25} + 6024w^{26} + 31\,840w^{27} + 10\,393w^{28} + 20\,669w^{29} \\ &+ 4477w^{30} + 29\,192w^{31} + 20\,075w^{32} + 2957w^{33} + 2003w^{34}. \end{aligned} \tag{74}$$

Although we have obtained such polynomials for the *four primes*, previously mentioned, our attempted rational reconstruction [24–26] of the exact $P(w)$ was not successful. There is no further information to guide our quest for the exact $P(w)$. There are only two indicial exponents (5 and −34) corresponding to the two points $w = 0$ and $w = \infty$, respectively. The linear differential operator $\tilde{L}_1$ is a first-order linear differential operator of the form

$$\tilde{L}_1 = \frac{\mathrm{d}}{\mathrm{d}w} + \frac{\mathrm{d}\ln(P(w))}{\mathrm{d}w}. \tag{75}$$

It is thus automatically globally nilpotent. Global nilpotence is a very severe constraint to fulfil for higher order linear differential operators, but for first-order operators like (75) it provides no additional constraints on $P(w)$.

6. Comments and speculations

In view of the previous results, we give some concluding remarks. The linear differential operator L_{29}, corresponding to $\tilde{\chi}^{(5)} - \tilde{\chi}^{(3)}/2 + \tilde{\chi}^{(1)}/120$, can be written as

$$L_{29} = L_5 \cdot L_{13} \cdot L_{11}, \tag{76}$$

with

$$L_{11} = (Z_2 \cdot N_1) \oplus V_2 \oplus (F_3 \cdot F_2 \cdot L_1^s), \tag{77}$$

$$L_{13} = L_{12} \cdot \tilde{L}_1. \tag{78}$$

The linear differential operator L_5 is equivalent to the symmetric fourth power of L_E. This linear differential operator is the factor of maximum order, assuming that the factorization scheme of L_{12} is correct. The scheme of factorization (76) then generalizes what we have obtained for $\tilde{\chi}^{(3)}$ and $\tilde{\chi}^{(4)}$.

[25] There are not enough logarithmic solutions at the other singular points.
[26] With the solution $P(w)$, we have the coefficients for the deepest combination series $w^5 + a_6w^6 + a_7w^7 + a_8w^8 + a_9w^9 + \cdots$.

Our conjecture on the structure of the $\tilde{\chi}^{(n)}$, namely, (23) and (24) would give, for the six-particle contribution $\tilde{\chi}^{(6)}$, the following scheme:

$$\tilde{\chi}^{(6)} = \alpha_4 \cdot \tilde{\chi}^{(4)} + \alpha_2 \cdot \tilde{\chi}^{(2)} + \Phi^{(6)}, \tag{79}$$

with $\Phi^{(6)}$ a solution of a linear ODE of order q whose corresponding linear differential operator factorizes as $L_q = L_6 \cdot L_{q-6}$, and where the left-factor L_6 is equivalent to the symmetric fifth power of L_E.

As far as the singularities are concerned, the 11th-order linear differential operator L_{11} has only the singularities of the linear ODE corresponding to L_7 (the operator for $\tilde{\chi}^{(3)}$) and the 'unknown'[27] $w = 1/2$. This singularity $w = 1/2$ occurs in the third-order linear differential operator F_3. The second-order differential operator F_2 is *responsible for the* $\rho = -5/4$, $\rho = -7/4$ *singular behaviour* around the ferromagnetic point $w = 1/4$ (see table 4 in [7]).

The singularities of the linear ODE corresponding to the block L_{12} are (besides $w = 0, \pm 1/4$):

$$\begin{aligned}
&(1-w)(1+2w)(1+3w+4w^2)\\
&(1+w)(1-3w+w^2)(1+2w-4w^2)(1-w-3w^2+4w^3)\\
&(1+8w+20w^2+15w^3+4w^4)(1-7w+5w^2-4w^3)\\
&(1+4w+8w^2).
\end{aligned} \tag{80}$$

The singularities in the first line are also singularities of the linear ODE for $\tilde{\chi}^{(3)}$. Note that at $w = -1$, $w = 1$ and $w = -1/2$, for instance, the linear differential equation corresponding to L_{12} has logarithmic solutions. Therefore, at least one of the factors (if the scheme is correct) in the block L_{12} is not equivalent to a symmetric power of L_E. If we consider the possibility that the linear differential operator of order 12 L_{12} is irreducible, this would mean that we are faced with a highly restricted object, which is globally nilpotent and not equivalent[28] to the symmetric eleventh power of L_E.

Let us close with the following question arising from some of the modular calculations and rational reconstructions presented in this paper. Is it in general easier to generate series for many primes, use these to reconstruct the exact series and hence obtain the exact linear ODE, or is it easier to obtain the linear ODE for a smaller number of primes and then carry out the rational reconstruction on the coefficients of the linear ODE? To disabuse the reader of the obvious first impression that the second method must be easier, we would like to point out that when we opt for a non-minimal order linear ODE, we gain by way of a reduction in the number of terms necessary to find the linear ODE, but this comes at a cost of an increase in the size of the coefficients in the linear ODE (see the last paragraph of Appendix A in [7] for an estimate for $\tilde{\chi}^{(3)}$). Even if we are dealing with the minimal order linear ODE, the coefficients in the right factors have fewer digits than the coefficients occurring in the left factors. For instance, considering the first factorization in (39), the maximum number of digits is 6 in Z_2, it increases to 27 for Y_3 and to 33 for M_1.

7. Conclusion

Using the Fuchsian linear ODE of $\tilde{\chi}^{(5)}$ (obtained modulo a single prime p_r), we have been able to go quite a long way towards understanding the factorization of its corresponding (minimal

[27] Unknown with respect to the $\Phi_H^{(5)}$ integrals [9] and to our Landau singularity analysis [7].

[28] But might, for instance, be equivalent to a symmetric power of a smaller order globally nilpotent operator related to modular forms.

order) linear differential operator L_{33}. In particular we have found several quite remarkable results.

The direct-sum structure of L_{33} generalizes what we have found for the linear differential operators of $\tilde{\chi}^{(3)}$ and $\tilde{\chi}^{(4)}$. In the linear differential operators of $\tilde{\chi}^{(5)}$ we found not only the occurrence of a term proportional to $\tilde{\chi}^{(3)}$, but also the occurrence of a term proportional to $\tilde{\chi}^{(1)}$. We conjecture that this structure occurs for all $\tilde{\chi}^{(n)}$, i.e. we expect to see terms in $\tilde{\chi}^{(n)}$ proportional to $\tilde{\chi}^{(n-2k)}$.

The linear differential operator L_{29} annihilating the 'depleted' linear combination (14) for $\tilde{\chi}^{(5)}$ follows the same structure seen for $\tilde{\chi}^{(3)}$ and $\tilde{\chi}^{(4)}$. The left-most factor of L_{29} is equivalent to the symmetric fourth power of the second-order operator corresponding to complete elliptic integrals of the first (or second) kind.

Some right factors of L_{29} are given in exact arithmetic. In particular one notes the occurrence of a very simple second-order operator V_2 and of the remarkable factor Z_2 that occurred in $\tilde{\chi}^{(3)}$. Using the series of $\tilde{\chi}^{(5)}$ obtained for p_r as well as three additional primes, we have obtained the linear ODE modulo of each prime and checked that the mentioned right factors are indeed exact. We have used the results for the four primes to see whether a rational reconstruction of right factors is feasible.

Two of the right factors, F_2 and F_3 (of order two and three, respectively), are highly restricted *globally nilpotent* linear differential operators, but, unfortunately, we have not been able to find exact solutions as we did for the linear differential operator Z_2 occurring in the factorization of the linear differential operator for $\tilde{\chi}^{(3)}$. Providing a better understanding of these operators, say in terms of *modular forms*, is clearly a natural (but actually quite difficult) challenge.

The incomplete part of our analysis is concerned with the 13th-order linear differential operator L_{13}. For this operator we did find a right factor of order one which, *quite remarkably, has a polynomial solution*. The factorization of the remaining 12th-order linear differential L_{12} is beyond our current computational resources. By producing all the 12 formal solutions of L_{12}, a factorization scheme appears where the differential operator L_{12} (if reducible) could factorize into three fourth-order operators, with a possible scenario that one of the fourth-order operators could factor into two second-order operators. Clearly some work remains to be done to better understand L_{12}, and hopefully find the exact fourth-order operators in its factorization. We thus hope to gain a better understanding of their very nature (are they symmetric powers of L_E or perhaps linear ODEs associated with modular forms, namely hypergeometric functions with a Hauptmodul pull-back).

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Appendix A. Solution of the order four linear ODE M_2 occurring in $\tilde{\chi}^{(4)}$

Defining $x = w^2$ and

$$K = {}_2F_1([1/2, 1/2], [1], 16x), \qquad E = {}_2F_1([1/2, -1/2], [1], 16x) \tag{A.1}$$

the solution (analytical at $x = 0$) of the linear ODE corresponding to M_2 is

$$\frac{1}{x^4 \cdot (1-16x)^4(1-4x)(7+80x)} \cdot ((1-16x) \cdot P_{3,0} \cdot K^3 - 3 \cdot P_{2,1} \cdot K^2E \\ - 3 \cdot P_{1,2} \cdot KE^2 - 3 \cdot P_{0,3} \cdot E^3)$$

with

$$P_{3,0} = 819\,200x^5 - 1050\,624x^4 + 494\,976x^3 - 39\,128x^2 - 90x + 63,$$
$$P_{2,1} = 26\,214\,400x^6 + 1458\,176x^5 - 4698\,112x^4 + 678\,464x^3 - 26\,120x^2 - 818x + 63,$$
$$P_{1,2} = 7208\,960x^5 + 1169\,408x^4 - 300\,288x^3 + 8728x^2 + 538x - 63,$$
$$P_{0,3} = 363\,520x^4 + 53\,696x^3 - 1144x^2 - 86x + 21.$$

Appendix B. The ODE formula

The linear differential equations annihilating a series $S(x)$ we are interested in are Fuchsian. This means that all singular points of the linear ODE, and in particular $x = 0$ and $x = \infty$, are regular. A form of the linear differential operator that automatically satisfies this constraint is

$$L_{QD} = \sum_{i=0}^{Q} \left(\sum_{j=0}^{D} a_{ij} \cdot x^j \right) \cdot \left(x \frac{\mathrm{d}}{\mathrm{d}x} \right)^i, \qquad a_{Q0} \neq 0, \quad a_{QD} \neq 0. \tag{B.1}$$

The condition $a_{Q0} \neq 0$ (respectively $a_{QD} \neq 0$) is required to make $x = 0$ (respectively $x = \infty$) a regular singular point.

Note that it is the use of the (homogeneity[29]) operator $x\frac{\mathrm{d}}{\mathrm{d}x}$ (rather than just $\mathrm{d}/\mathrm{d}x$), which leads to the above conditions guaranteeing the regularity of the singular points $x = 0$ and $x = \infty$ and to the equality of the degrees of the polynomials in front of the derivatives. For the operator $\mathrm{d}/\mathrm{d}x$, a simple rearrangement of terms shows that (B.1) can be re-written as

$$L_{QD} = \sum_{i=0}^{Q} \left(\sum_{j=0}^{D} b_{ij} \cdot x^{j+i} \right) \cdot \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^i, \tag{B.2}$$

where the coefficients b_{ij} are linear combinations of a_{ij}. This is the form of the Fuchsian linear ODE we have used in many previous papers (e.g., [8–11, 27, 28]). The Fuchsian character of the ODE is reflected in the decreasing degrees of the successive polynomials in front of the derivatives.

To find the linear ODE annihilating $S(x)$, the coefficients a_{ij} in (B.1) have to be determined. This can be done by demanding $L_{QD}(S(x)) = 0$, resulting in a set of linear equations for the unknown coefficients a_{ij}. In [7] this set of linear equations was put into a well-defined order and if the corresponding $N_{QD} \times N_{QD}$ determinant (with $N_{QD} = (Q+1)(D+1)$) vanishes, a non-trivial solution exists. The zero-determinant condition was checked by creating an upper triangular matrix U using standard Gaussian elimination and a solution exists if we find $U(N, N) = 0$ for some N. The N for which $U(N, N) = 0$ is thus the minimum number

[29] Also called Euler's operator. Recall that $\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \cdot x^k = k^n x^k$.

Table B1. ODE formula for $\tilde{\chi}^{(n)}$, $n = 1, 2, \ldots, 5$ and for the combinations $6\tilde{\chi}^{(n+2)} - n\tilde{\chi}^{(n)}$, $n = 1, 2, 3$. The last column gives the value of the parameter f corresponding to the same Q and D considered in [7].

Series	$dQ + qD - C$	Q	D	f
$\tilde{\chi}^{(1)}$	$1Q + 1D + 1$	1	1	1
$\tilde{\chi}^{(2)}$	$1Q + 2D + 1$	2	1	1
$\tilde{\chi}^{(3)}$	$12Q + 7D - 37$	11	17	2
$\tilde{\chi}^{(4)}$	$7Q + 10D - 36$	15	9	1
$\tilde{\chi}^{(5)}$	$72Q + 33D - 887$	56	129	8
$6\tilde{\chi}^{(3)} - \tilde{\chi}^{(1)}$	$12Q + 6D - 26$	10	17	2
$6\tilde{\chi}^{(4)} - 2\tilde{\chi}^{(2)}$	$6Q + 8D - 17$	13	8	1
$6\tilde{\chi}^{(5)} - 3\tilde{\chi}^{(3)}$	$68Q + 30D - 732$	52	120	9

of coefficients needed to find the linear ODE for given Q and D. The deviation between the actual number of coefficients needed N, and the generic (maximum) $(Q+1)(D+1)$ was called Δ in [7].

To fully understand the deviation $\Delta = (Q+1)(D+1) - N$, we may alternatively compute the nullspace of the matrix U. The dimension of the nullspace, if a solution exists, is related to Δ. In others words, solving $L_{QD}\,(S(x)) = 0$ term by term will fix all the coefficients but leaves f coefficients unfixed among the N_{QD} ones. These are all independent ODE solutions for given Q and D.

In [7] we reported a remarkable formula arising from empirical observation

$$N = d \cdot Q + q \cdot D - C = (Q+1)(D+1) - f, \tag{B.3}$$

where we have replaced the parameter Δ used in [7] by the parameter f that we can now understand *as the number of independent solutions* for given Q and D (this understanding will be useful later). The ODE formula (B.3) should be understood as follows: for a long series $S(x)$ we use three (or more) sets of Q and D and solve $L_{QD}\,(S(x)) = 0$ (by nullspace or term by term). From this we obtain the value of the parameter f (if $f > 0$, otherwise we increase Q and/or D) for each pair (Q, D). These values (Q, D, f) are then used to determine d, q and C in (B.3). In all cases we have investigated, the parameter q is the order of the minimal order linear ODE that annihilates $S(x)$. The parameter d is the number of singularities (counted with multiplicity) excluding any apparent singularities and the 'true' singular point $x = 0$ which is already taken care of by the use of the differential operator $x\frac{\mathrm{d}}{\mathrm{d}x}$.

We revisit in table B1 some ODE formulae from table B1 of [7]. We give the value of the parameter f corresponding to the same Q and D considered in [7]. The first observation is that, generally, both ODE formulae (in table B1 and in [7]) agree. When they do not, the difference is in the parameter C. But we remark that $C - f$ always equals $C - \Delta$, which is easily understood from the equality in (B.3). The second observation is that, for the linear ODE which have the constant as solution (i.e. $\tilde{\chi}^{(4)}$ and $6\tilde{\chi}^{(4)} - 2\tilde{\chi}^{(2)}$), the parameter q appears as the actual one.

With the nullspace computation we now understand the constant f (Δ_0 in [7]). Thus for the minimal order ODE, one should have $f = 1$ since the minimal order ODE is unique. Setting $Q = q$ and $D = d + D_{\mathrm{app}}$, where D_{app} is the degree of the polynomial whose roots are apparent singularities, one obtains the *exact relation*

$$D_{\mathrm{app}} = (d-1)(q-1) - C - 1 \tag{B.4}$$

between the constant C and the degree $D_{\rm app}$. For $\tilde{\chi}^{(3)}$ one has $d = 12, q = 7$ and $C = 37$ giving $D_{\rm app} = 28$ which is [10, 12] the degree of the polynomial carrying apparent singularities in the linear ODE of $\tilde{\chi}^{(3)}$. For $\tilde{\chi}^{(4)}$ one has $d = 7, q = 10$ and $C = 36$ giving $D_{\rm app} = 17$, which is [11] the degree of the polynomial carrying apparent singularities in the ODE of $\tilde{\chi}^{(4)}$. Similarly for $\tilde{\chi}^{(5)}$, with $d = 72, q = 33$ and $C = 887$ we obtain the degree of the apparent polynomial $D_{\rm app} = 1384$, which is in agreement with what appears in the linear ODE for $\tilde{\chi}^{(5)}$ reduced to its minimal order.

Note also that (B.4) is valid for linear ODEs without an apparent polynomial ($\tilde{\chi}^{(1)}$ and $\tilde{\chi}^{(2)}$). But there are cases where the parameter C is negative while the linear ODE has an apparent polynomial. This is the case we consider now.

B.1. The ODE formula for the factors

We first show how the apparent polynomials occur in a factorization of linear differential operators such as

$$\mathcal{L} = L \cdot R,$$

where the factors L and R are monic[30] and of minimal order, denoted respectively q_L and q_R. Denoting by $P_{\rm app}$, the apparent polynomial occurring in $\mathcal{L}$, one knows that this polynomial should appear as an apparent polynomial in the left-operator L. It may happen that the right-operator R also contains a polynomial Q of apparent singularities and this polynomial *should not appear in* $\mathcal{L}$. For this to happen, the left-operator L must have the roots of Q as true singularities. Furthermore, Q should occur in L to the power of the order of L, i.e. as Q^{q_L}. The local exponents for L at any root of Q are $-1, 1, 2, \ldots, q_L - 1$. If we remove the singularity Q^{-1} from L, the new linear differential operator $\tilde{L}$ will have Q as an *apparent polynomial* and will occur as Q^{q_L-1} with local exponents $0, 2, 3, \ldots, q_L$.

Consider, as an example, the series S for $\tilde{\chi}^{(3)}$ annihilated by a seventh-order linear ODE with L_7 as the corresponding linear differential operator. We know that this operator factorizes as (among other factorizations (39))

$$L_7 = L \cdot R = (M_1 \cdot Y_3) \cdot (Z_2 \cdot N_1). \tag{B.5}$$

Assume that the right-operator R is known. The aim is to produce the left-operator L by acting on S with R. The series $R(S)$ will satisfy a linear ODE corresponding to L.

For the right-operator $R = Z_2 \cdot N_1$, the left-hand side of the ODE formula (B.3) reads

$$d_R \cdot Q + q_R \cdot D - C_R = 8Q + 3D - 9. \tag{B.6}$$

Putting these values into (B.4) we obtain $D^R_{\rm app} = 4$ which is the degree of the apparent polynomial Q occurring in $R = Z_2 \cdot N_1$.

The linear ODE for the left-operator L produced from the series $R\,(S)$ when R is taken *monic and of minimal order*, has the ODE formula

$$d_L \cdot Q + q_L \cdot D - C_L = 15Q + 4D - 1. \tag{B.7}$$

The degree of the apparent polynomial for $L = M_1 \cdot Y_3$, computed by (B.4), is $D^L_{\rm app} = 40$, which is the degree of $P_{\rm app}$ (the apparent polynomial of L_7, see above) plus three times the degree $D^R_{\rm app}$, and we still have the roots of Q appearing with multiplicity one in $d_L = 15$.

In computations modulo a prime, and for high order linear ODEs, it is obvious that it is easier to work with non-monic operators. This results in removing the pole part of the polynomial Q, leaving its apparent part in the left-operator L.

[30] Normalization of the head polynomial of the linear differential operator.

As an example, we will reproduce the series $R\ (S)$ with R non-monic but still of minimal order. The left-hand side of the ODE formula (B.3), corresponding to L, reads

$$d_L \cdot Q + q_L \cdot D - C_L = 4Q + 4D + 32. \tag{B.8}$$

From (B.4) we obtain the degree of the apparent singularities in L as $40 = 28 + 3 \times 4$. Furthermore, recalling the left-hand side of the ODE formula (B.3) (see table B1)

$$dQ + qD - C = 12Q + 7D - 37 \tag{B.9}$$

for the full L_7, one has $d = d_R + d_L$ and

$$C = C_L + \frac{q - q_R - 1}{q_R - 1} \cdot C_R + \frac{q_R}{q_R - 1} \cdot \big((q - q_R - 1)D_{\rm app}^R + dq_R - 2q_R + q - d\big). \tag{B.10}$$

Remark. Even if the various parameters in the ODE formula are now understood, we should recognize that we still do not know how this ODE formula can be proved, nor where it comes from. The various formulae dealing with the apparent polynomial degree (in fact upper bounds, e.g. [29–31]) in Fuchsian linear ODEs introduce ingredients that go beyond our experimental mathematics framework.

Appendix C. Some linear differential operators in exact arithmetic

The linear differential operators V_2, F_2 and F_3 occurring in the decomposition of L_{11}

$$L_{11} = (Z_2 \cdot N_1) \oplus V_2 \oplus \big(F_3 \cdot F_2 \cdot L_1^s\big), \tag{C.1}$$

read respectively

$$V_2 = {\rm D}_w^2 - \frac{(3 + 8w + 16w^2)}{(1 + 4w)(1 - 4w)w} \cdot {\rm D}_w + 4\frac{1 + 7w + 4w^2}{(1 - 4w)(1 + 4w)^2w^2}, \tag{C.2}$$

and

$$F_2 = {\rm D}_w^2 - \frac{P_1}{P_2} \cdot {\rm D}_w - \frac{P_0}{P_2}, \tag{C.3}$$

with[31]

$$\begin{aligned}
P_2 &= (1 - 4w) \cdot p_2,\\
p_2 &= w \cdot (1 - 4w)(1 + 4w)(1 - w)(1 + 2w)(1 + 3w + 4w^2)\\
&\quad \times (1 + w - 24w^2 - 145w^3 - 192w^4 + 96w^5 + 128w^7),\\
P_1 &= (1 - w)(1 - 4w)(1 + 2w)(40\,960w^{11} + 24\,576w^{10} + 51\,712w^9 - 66\,816w^8\\
&\quad - 138\,176w^7 - 88\,704w^6 - 29\,940w^5 - 5394w^4 - 272w^3 + 92w^2 + 11w + 1),\\
P_0 &= 262\,144w^{13} - 65\,536w^{12} + 335\,872w^{11} - 934\,912w^{10} - 743\,424w^9 + 703\,488w^8\\
&\quad + 867\,776w^7 + 371\,848w^6 + 96\,744w^5 + 14\,710w^4 + 2144w^3 + 398w^2 + 9w - 11.
\end{aligned}$$

It is possible to get rid of the apparent singularities occurring in F_2, by multiplying F_2 at the left, by a first-order linear differential operator $\mathcal{L}_1$,

$$\mathcal{L}_1 = {\rm D}_w - \frac{1}{73\,326\,885\,520} \cdot \left(\frac{q_0}{p_2} + \frac{256\,352\,914\,629}{1 + 3w + 4w^2}\right),$$

$$\text{with :} \qquad \frac{q_0}{p_2} = \frac{\rm d}{{\rm d}w}\ln(R(w)), \tag{C.4}$$

[31] Note that the factors $(1 + 2w)$ and $(1 - w)$ appear to the power one in both P_2 and P_1. Linear differential operators can be Fuchsian without having descending powers of the factors giving rise to the singularities.

where $R(w)$ is a rational function (with integer coefficients) and

$$q_0 = 244\,820\,905\,584 + 1372\,135\,276\,587w + 1384\,232\,623\,846w^2 - 13\,621\,658\,367\,235w^3 \\ - 150\,856\,196\,156\,313w^4 - 1054\,439\,469\,518\,747w^5 - 3472\,090\,747\,016\,314w^6 \\ - 3873\,078\,043\,825\,712w^7 + 3114\,022\,565\,962\,720w^8 + 12\,058\,813\,946\,690\,432w^9 \\ + 10\,882\,841\,933\,451\,520w^{10} - 1075\,293\,814\,167\,552w^{11} - 3544\,662\,480\,211\,968w^{12} \\ - 9348\,606\,615\,093\,248w^{13},$$

thus yielding a third-order desingularized Fuchsian operator. This is the so-called 'desingularization' procedure which preserves the Fuchsian character of the linear differential operators. Note however that the desingularization procedure *does not* preserve the remarkable property of global nilpotence of the highly restricted second-order differential operator F_2. The new desingularized third-order differential operator is *no longer globally nilpotent* because the first-order differential operator $\mathcal{L}_1$ is *not globally nilpotent*. The breaking of global nilpotence comes from the factor $256352914629/(1 + 3w + 4w^2)$ in (C.4) which is not a logarithmic derivative of a rational function.

Next we focus on the 'physical' singularity $w = 1/4$. One can change the operator F_2 into a slightly simpler one as follows:

$$F_2 \quad\longrightarrow\quad \tilde{F}_2 = F_2 \cdot (1 - 4w)^{-11/4}, \tag{C.5}$$

where the dot corresponds to a multiplication of (differential) operators. It is important to note that the solutions of $\tilde{F}_2$ around the 'physical' singularity $w = 1/4$ are in fact Puiseux series in $u = (w - 1/4)^{1/2}$. In other words $\tilde{F}_2$ rewritten in terms of the variable u is *not* singular at $w = 1/4$.

The calculations performed on Z_2 yielded a modular form interpretation of Z_2 (see [6]). A crucial step corresponded to discovering the covering

$$w \quad\longrightarrow\quad t = \frac{-8w}{(1-w)(1-4w)},$$
$$\text{or}\quad Q(t, w) = 0, \qquad \text{with}\quad Q(t, w) = 4t \cdot w^2 - (5t - 8) \cdot w + t, \tag{C.6}$$

which wraps the singularities of Z_2 onto the three singularities of ${}_2F_1$, namely $0, 1, \infty$. Do note that the apparent polynomial for $\tilde{Z}_2$ occurs as a vanishing condition of the discriminant in w of the covering polynomial $Q(t, w)$

$$\mathrm{discrim}(Q(t, w), w) = (9t - 8)(t - 8) = 8\frac{(1 - 2w)^2}{(1 - w)(1 - 4w)}. \tag{C.7}$$

Trying to perform a similar calculation for F_2 in order to discover some *modular form interpretation* for F_2, we observe that it is not straightforward to find a covering, such as (C.6), wrapping all the singularities of F_2 onto 0, 1 and ∞, and such that the discriminant in w (like (C.7)) of the corresponding covering polynomial $Q(t, w)$, could correspond to the quite involved apparent polynomial of F_2, namely $1 + w - 24w^2 - 145w^3 - 192w^4 + 96w^5 + 128w^7$. For these reasons we have failed in finding a modular form interpretation of the highly restricted linear differential operator F_2.

The third-order linear differential operator F_3 reads

$$F_3 = \mathrm{D}_w^3 + (1 + 2w)P_s^2\frac{P_2}{P_3} \cdot \mathrm{D}_w^2 + 2P_s\frac{P_1}{P_3} \cdot \mathrm{D}_w + \frac{P_0}{P_3}, \tag{C.8}$$

where

$$P_s = -w \cdot (1 - 4w)(1 + 4w)(1 + w - 24w^2 - 145w^3 - 192w^4 + 96w^5 + 128w^7),$$
$$P_3 = (1 - w)(1 - 2w)(1 + 3w + 4w^2)(1 + 2w)^2 \cdot P_s^3 \cdot p_3,$$

$$p_3 = 5629\,499\,534\,213\,120w^{37} + 5348\,024\,557\,502\,464w^{36} - 62\,874\,472\,922\,742\,784w^{35}$$
$$+ 339\,080\,589\,913\,096\,192w^{34} + 132\,348\,214\,635\,397\,120w^{33} + 354\,600\,746\,294\,968\,320w^{32}$$
$$+ 1383\,732\,497\,338\,073\,088w^{31} – 269\,118\,080\,922\,157\,056w^{30} – 1021\,414\,905\,992\,970\,240w^{29}$$
$$+ 401\,943\,021\,895\,024\,640w^{28} + 378\,516\,473\,892\,569\,088w^{27} - 379\,126\,125\,978\,189\,824w^{26}$$
$$- 181\,955\,521\,970\,962\,432w^{25} + 182\,991\,453\,503\,356\,928w^{24} + 119\,809\,766\,351\,437\,824w^{23}$$
$$- 34\,528\,714\,733\,649\,920w^{22} - 46\,719\,523\,456\,286\,720w^{21} - 1865\,897\,472\,688\,128w^{20}$$
$$+ 9861\,412\,040\,736\,768w^{19} + 1690\,374\,175\,916\,032w^{18} - 1285\,664\,678\,690\,816w^{17}$$
$$- 304\,716\,171\,767\,808w^{16} + 112\,170\,181\,177\,344w^{15} + 30\,517\,814\,178\,816w^{14}$$
$$- 7815\,766\,123\,264w^{13} - 2274\,047\,571\,904w^{12} + 456\,062\,896\,896w^{11}$$
$$+ 150\,282\,885\,872w^{10} - 10\,690\,267\,808w^{9} - 6048\,942\,832w^{8} - 486\,602\,112w^{7}$$
$$+ 33\,772\,908w^{6} + 25\,075\,632w^{5} + 4670\,454w^{4} + 13\,440w^{3} - 69\,066w^{2} - 5169w - 63,$$

$$P_2 = 2582\,544\,170\,319\,337\,226\,240w^{51} + 2029\,141\,848\,108\,050\,677\,760w^{50}$$
$$- 32\,885\,932\,997\,405\,703\,143\,424w^{49} + 193\,641\,813\,610\,004\,500\,971\,520w^{48}$$
$$+20\,426\,022\,066\,743\,356\,162\,048w^{47} + 288\,714\,242\,895\,676\,430\,090\,240w^{46}$$
$$+618\,280\,187\,651\,267\,892\,346\,880w^{45} - 648\,919\,373\,873\,770\,257\,186\,816w^{44}$$
$$- 863\,129\,472\,633\,247\,214\,075\,904w^{43} - 1021\,011\,939\,308\,518\,347\,112\,448w^{42}$$
$$- 220\,333\,306\,036\,159\,265\,112\,064w^{41} + 1659\,564\,100\,832\,816\,225\,320\,960w^{40}$$
$$+588\,473\,220\,873\,831\,600\,619\,520w^{39} - 1065\,067\,759\,683\,713\,707\,802\,624w^{38}$$
$$- 9030\,793\,760\,523\,344\,150\,528w^{37} + 805\,481\,511\,795\,301\,371\,871\,232w^{36}$$
$$- 122\,169\,749\,668\,787\,845\,595\,136w^{35} - 629\,129\,357\,422\,714\,417\,053\,696w^{34}$$
$$- 87\,120\,833\,646\,056\,343\,339\,008w^{33} + 304\,015\,333\,576\,904\,250\,753\,024w^{32}$$
$$+ 143\,209\,349\,380\,404\,068\,483\,072w^{31} - 67\,135\,556\,652\,765\,458\,464\,768w^{30}$$
$$- 68\,161\,001\,548\,708\,224\,958\,464w^{29} - 1506\,006\,178\,531\,414\,900\,736w^{28}$$
$$+15\,819\,782\,847\,593\,648\,750\,592w^{27} + 4086\,678\,104\,179\,764\,363\,264w^{26}$$
$$- 1909\,688\,698\,451\,711\,754\,240w^{25} - 970\,204\,468\,920\,909\,561\,856w^{24}$$
$$+ 94\,919\,087\,350\,092\,267\,520w^{23} + 123\,918\,740\,818\,141\,650\,944w^{22}$$
$$+ 4687\,965\,654\,930\,399\,232w^{21} - 10\,707\,547\,611\,722\,045\,440w^{20}$$
$$- 1144\,659\,629\,046\,790\,144w^{19} + 806\,082\,415\,949\,659\,264w^{18}$$
$$+ 149\,774\,462\,467\,091\,328w^{17} - 48\,096\,268\,859\,594\,016w^{16}$$
$$- 16\,578\,560\,990\,131\,776w^{15} + 424\,243\,043\,096\,032w^{14} + 905\,149\,437\,225\,280w^{13}$$
$$+ 139\,111\,711\,404\,072w^{12} - 7972\,709\,043\,232w^{11} - 6405\,062\,530\,332w^{10}$$
$$- 1037\,367\,028\,148w^{9} + 6971\,928\,216w^{8} + 30\,288\,912\,150w^{7}$$
$$+ 3873\,954\,375w^{6} - 115\,755\,798w^{5} - 60\,227\,304w^{4}$$
$$- 3099\,678w^{3} + 211\,068w^{2} + 21\,432w + 315,$$

$$P_1 = 19\,267\,255\,250\,108\,152\,471\,879\,680w^{61} + 26\,483\,031\,235\,932\,970\,358\,407\,168w^{60}$$
$$- 256\,308\,802\,040\,991\,428\,795\,957\,248w^{59} + 1579\,949\,167\,665\,869\,307\,621\,933\,056w^{58}$$
$$+ 650\,374\,789\,771\,441\,405\,855\,531\,008w^{57} + 5216\,643\,706\,804\,247\,528\,946\,532\,352w^{56}$$
$$+ 7917\,834\,014\,591\,751\,323\,461\,353\,472w^{55} - 3351\,835\,287\,019\,392\,824\,172\,871\,680w^{54}$$

$$
\begin{aligned}
&- 11\,064\,588\,131\,657\,140\,234\,556\,014\,592w^{53} - 34\,985\,695\,129\,493\,606\,924\,629\,835\,776w^{52}\\
&- 27\,150\,264\,881\,506\,217\,380\,601\,135\,104w^{51} + 21\,916\,196\,537\,425\,570\,804\,428\,439\,552w^{50}\\
&+ 42\,001\,979\,686\,686\,526\,227\,299\,172\,352w^{49} + 25\,840\,385\,187\,494\,624\,677\,295\,292\,416w^{48}\\
&- 5424\,492\,229\,252\,674\,644\,950\,908\,928w^{47} - 17\,794\,118\,224\,994\,570\,424\,773\,771\,264w^{46}\\
&+ 2915\,867\,386\,820\,035\,753\,799\,581\,696w^{45} + 7186\,426\,242\,807\,565\,546\,487\,283\,712w^{44}\\
&- 14\,774\,359\,259\,974\,734\,620\,101\,967\,872w^{43} - 14\,907\,706\,789\,958\,430\,400\,446\,464\,000w^{42}\\
&+ 8071\,574\,290\,338\,795\,697\,467\,293\,696w^{41} + 15\,504\,536\,229\,837\,797\,153\,841\,348\,608w^{40}\\
&+ 2626\,569\,595\,260\,883\,926\,907\,879\,424w^{39} - 6860\,129\,397\,955\,391\,680\,596\,148\,224w^{38}\\
&- 4292\,305\,256\,193\,524\,038\,115\,524\,608w^{37} + 797\,725\,481\,517\,011\,621\,914\,869\,760w^{36}\\
&+ 1720\,922\,858\,808\,986\,948\,924\,866\,560w^{35} + 390\,533\,143\,866\,840\,910\,481\,326\,080w^{34}\\
&- 290\,442\,221\,202\,989\,562\,927\,775\,744w^{33} - 165\,938\,014\,975\,046\,925\,940\,686\,848w^{32}\\
&+ 9585\,161\,397\,342\,427\,263\,533\,056w^{31} + 28\,502\,663\,270\,123\,757\,533\,921\,280w^{30}\\
&+ 4489\,386\,799\,471\,924\,718\,338\,048w^{29} - 2717\,486\,608\,897\,256\,297\,267\,200w^{28}\\
&- 908\,343\,774\,367\,384\,075\,960\,320w^{27} + 162\,310\,791\,240\,979\,996\,000\,256w^{26}\\
&+ 103\,239\,269\,328\,878\,845\,726\,720w^{25} - 7750\,369\,303\,138\,783\,333\,376w^{24}\\
&- 11\,489\,552\,784\,013\,679\,223\,808w^{23} - 712\,249\,616\,867\,767\,788\,544w^{22}\\
&+ 991\,748\,945\,187\,237\,072\,640w^{21} + 252\,693\,598\,182\,584\,513\,344w^{20}\\
&- 21\,130\,273\,995\,450\,588\,928w^{19} - 20\,284\,808\,101\,979\,844\,832w^{18}\\
&- 3056\,348\,368\,556\,274\,592w^{17} + 345\,270\,164\,930\,943\,040w^{16}\\
&+ 205\,893\,879\,174\,875\,432w^{15} + 28\,654\,368\,006\,663\,856w^{14}\\
&- 2030\,520\,435\,693\,824w^{13} - 1374\,304\,588\,556\,840w^{12} - 166\,988\,492\,206\,488w^{11}\\
&+ 12\,760\,292\,849\,076w^{10} + 5484\,990\,319\,472w^{9} + 367\,504\,601\,004w^{8}\\
&- 50\,197\,207\,920w^{7} - 9218\,315\,844w^{6} - 277\,909\,095w^{5}\\
&+ 48\,467\,763w^{4} + 5648\,070w^{3} + 265\,293w^{2} + 4620w - 63,
\end{aligned}
$$

$$
\begin{aligned}
P_0 = {}& 69\,634\,127\,209\,802\,640\,463\,075\,737\,600w^{70}\\
&+ 102\,981\,137\,018\,052\,571\,618\,170\,896\,384w^{69}\\
&- 1033\,960\,403\,593\,443\,123\,509\,300\,559\,872w^{68}\\
&+ 6716\,228\,494\,346\,939\,277\,472\,100\,777\,984w^{67}\\
&+ 830\,768\,383\,072\,984\,903\,026\,797\,969\,408w^{66}\\
&+ 34\,119\,483\,032\,722\,461\,380\,174\,390\,755\,328w^{65}\\
&+ 37\,151\,403\,895\,216\,351\,475\,147\,854\,577\,664w^{64}\\
&+ 7596\,402\,077\,224\,314\,128\,199\,487\,324\,160w^{63}\\
&- 57\,748\,765\,852\,096\,741\,713\,914\,269\,532\,160w^{62}\\
&- 309\,493\,302\,673\,497\,714\,830\,630\,511\,968\,256w^{61}\\
&- 232\,460\,008\,226\,528\,101\,141\,464\,649\,564\,160w^{60}\\
&+ 23\,931\,702\,098\,177\,545\,910\,680\,337\,514\,496w^{59}\\
&+ 427\,559\,960\,442\,089\,709\,631\,493\,273\,288\,704w^{58}\\
&+ 767\,305\,599\,958\,046\,596\,665\,201\,651\,613\,696w^{57}\\
&+ 238\,126\,263\,520\,324\,803\,598\,765\,665\,550\,336w^{56}
\end{aligned}
$$

$$
\begin{aligned}
&- 489\,368\,606\,355\,635\,167\,460\,530\,948\,407\,296w^{55}\\
&- 411\,394\,912\,009\,392\,610\,164\,715\,657\,625\,600w^{54}\\
&- 9190\,571\,673\,284\,172\,503\,536\,766\,025\,728w^{53}\\
&+ 4354\,675\,264\,071\,893\,610\,129\,679\,974\,400w^{52}\\
&- 197\,517\,596\,854\,575\,225\,398\,680\,040\,243\,200w^{51}\\
&- 100\,108\,534\,915\,833\,684\,237\,428\,566\,523\,904w^{50}\\
&+ 237\,797\,067\,305\,428\,725\,186\,438\,474\,235\,904w^{49}\\
&+ 297\,736\,260\,249\,409\,824\,018\,326\,167\,224\,320w^{48}\\
&+ 2827\,068\,864\,630\,764\,422\,668\,662\,865\,920w^{47}\\
&- 212\,811\,801\,685\,085\,801\,466\,392\,370\,741\,248w^{46}\\
&- 132\,552\,668\,920\,641\,958\,581\,606\,566\,330\,368w^{45}\\
&+ 36\,227\,143\,968\,974\,260\,317\,176\,152\,981\,504w^{44}\\
&+ 80\,360\,530\,046\,171\,440\,422\,025\,236\,054\,016w^{43}\\
&+ 25\,044\,389\,743\,118\,121\,276\,003\,435\,151\,360w^{42}\\
&- 16\,775\,611\,588\,713\,607\,092\,922\,243\,612\,672w^{41}\\
&- 14\,375\,299\,221\,388\,934\,261\,580\,907\,937\,792w^{40}\\
&- 797\,769\,185\,002\,542\,420\,001\,267\,122\,176w^{39}\\
&+ 3060\,062\,577\,366\,019\,941\,153\,762\,181\,120w^{38}\\
&+ 1048\,552\,246\,961\,478\,552\,732\,246\,736\,896w^{37}\\
&- 286\,300\,750\,377\,610\,217\,893\,186\,764\,800w^{36}\\
&- 238\,703\,363\,798\,670\,426\,041\,267\,257\,344w^{35}\\
&- 6453\,603\,219\,285\,212\,538\,454\,671\,360w^{34}\\
&+ 32\,067\,040\,600\,375\,745\,464\,846\,254\,080w^{33}\\
&+ 6256\,946\,928\,524\,452\,096\,094\,240\,768w^{32}\\
&- 3100\,756\,305\,550\,863\,462\,745\,636\,864w^{31}\\
&- 1271\,279\,614\,733\,459\,684\,395\,712\,512w^{30}\\
&+ 173\,570\,224\,057\,030\,875\,371\,798\,528w^{29}\\
&+ 187\,963\,836\,513\,544\,173\,604\,265\,984w^{28} + 20\,098\,454\,205\,158\,749\,911\,726\,080w^{27}\\
&- 15\,220\,510\,128\,449\,109\,866\,076\,160w^{26} - 5530\,208\,120\,756\,891\,132\,317\,696w^{25}\\
&- 31\,540\,092\,813\,892\,142\,535\,680w^{24} + 410\,046\,209\,080\,624\,124\,809\,344w^{23}\\
&+ 95\,695\,972\,411\,021\,353\,163\,264w^{22} - 3799\,963\,752\,408\,310\,388\,096w^{21}\\
&- 6052\,638\,686\,215\,258\,044\,992w^{20} - 1044\,163\,538\,474\,290\,733\,536w^{19}\\
&+ 69\,113\,265\,719\,111\,269\,072w^{18} + 55\,225\,911\,443\,186\,243\,360w^{17}\\
&+ 6988\,584\,609\,018\,020\,432w^{16} - 714\,608\,406\,420\,145\,560w^{15}\\
&- 313\,788\,688\,846\,958\,472w^{14} - 23\,383\,932\,527\,942\,400w^{13}\\
&+ 4392\,065\,243\,452\,176w^{12} + 942\,992\,856\,333\,120w^{11} + 18\,782\,060\,660\,376w^{10}\\
&- 11\,352\,161\,581\,890w^{9} - 1093\,090\,772\,088w^{8} + 23\,284\,774\,974w^{7}\\
&+ 9267\,369\,222w^{6} + 542\,276\,796w^{5} - 59\,916w^{4}\\
&- 3757\,362w^{3} - 465\,618w^{2} - 20\,622\,w - 126.
\end{aligned}
$$

We note that F_3 can be simplified as follows:

$$F_3 \quad \longrightarrow \quad \tilde{F}_3 = F_3 \cdot \frac{1}{\mu}, \tag{C.9}$$

where

$$\mu = w^2 \cdot (1-4w)^{9/2} \cdot (1+4w)^{7/2} \cdot (1-w) \cdot (1+2w)(1+3w+4w^2) \cdot \mathrm{App}(F_2), \tag{C.10}$$

where $\mathrm{App}(F_2)$ denotes the apparent polynomial for F_2, namely $1 + w - 24w^2 - 145w^3 - 192w^4 + 96w^5 + 128w^7$, and where the dot in (C.9) denotes the multiplication of (differential) operators. This just amounts to multiplying the solutions of F_3 by μ. Remarkably $\tilde{F}_3$ is *no longer singular* at $w = 1$ nor at the two roots of the quadratic $1 + 3w + 4w^2 = 0$. We find the following exponents at the remaining singularities:

$$\begin{array}{lll}
w = 0, & 0, 1, 3, & (\log^2), \\
w = 1/4, & 0, 1, 3/2, & \\
w = -1/4, & 0, 1, 5/2, & \\
w = \infty, & -18, -18, -16, & (\log^2), \\
w = 1/2, & 0, 1, 1/2, & \\
w = -1/2, & 0, 1, 1/2, & \\
\mathrm{App}(F_2) = 0, & 0, 2, 3. &
\end{array}$$

We have here an illustration of what we described in appendix B.1 where the third-order linear differential operator F_3 reads

$$F_3 = P_{\mathrm{sing}} \cdot \mathrm{App}(F_2)^3 \cdot \mathrm{App}(F_3) \cdot D_w^3 + \cdots, \tag{C.11}$$

where P_{sing} denote the 'true' singularity polynomial of F_3. We remark that the apparent polynomial of F_3 is the apparent polynomial appearing in the product $F_5 = F_3 \cdot F_2$. The polynomial $\mathrm{App}(F_2)$ is the apparent polynomial of F_2. It appears at the power of the order of F_3 for which it is a pole. When rescaled as done in $\tilde{F}_3$ the roots of $\mathrm{App}(F_2)$ become apparent singularities of $\tilde{F}_3$.

Note that the formal series of the linear differential operator $\tilde{F}_3$ are *Puiseux series* around all the singularities except $w = 0$ and $w = \infty$. These are *the only singular points around which $\tilde{F}_3$ has logarithmic solutions*. When the third-order operator $\tilde{F}_3$ is rewritten in terms of the variable $u = (w - w_s)^{1/2}$, where w_s is any singularity other than $w = 0$ or $w = \infty$, $\tilde{F}_3$ is *no longer singular* at w_s (in particular the ferromagnetic critical point $w = 1/4$ is *no longer* singular in the variable $u = (w - 1/4)^{1/2}$).

Appendix D. Experiment: rational reconstruction of the apparent polynomial in F_3

Write the linear differential operator F_3 as

$$F_3 = \mathcal{P}_3(w) P_{\mathrm{app}}(w^{37}) \cdot D_w^3 + \mathcal{P}_2(w) P_2(w^{51}) \cdot D_w^2 + \mathcal{P}_1(w) P_1(w^{61}) \cdot D_w + P_0(w^{70}),$$

where $\mathcal{P}_i(w)$ account for[32] the known multiplicities, and the argument w^n in the polynomials is used to show their respective degrees n. Assume that this linear ODE has been obtained for many primes. We want to carry out the rational reconstruction for each polynomial separately, basically because the polynomials at the lower derivatives are harder to obtain.

As it comes from our solver, the polynomial P_{app} cannot be reconstructed with nine primes. If we multiply all the mod prime coefficients by 2^{38} the rational reconstruction will

[32] These $\mathcal{P}_i(w)$'s are different from the ones in (62).

be successful with eight primes. If we multiply by 2^{50} the reconstruction succeeds with six primes. It should be noted that when the number of primes is not sufficient, the correctly reconstructed coefficients will be those of lower degrees or higher degrees depending on the magnitude of the scale used to multiply the coefficients. This then calls for a scaling of the variable itself. If we change the variable w to $w/2$ and multiply all coefficients by 2^{80}, the rational reconstruction is successful with just *five* primes. It is fortunate that the apparent polynomial is the easier polynomial to reconstruct. It will be used in further checks.

How can one guess the scaling (e.g. 2^{38} and 2^{80}) mentioned above? We have found that 2^{38} is the magnitude of the lower coefficient in $\mathcal{P}_3(w)$, which is an exactly known polynomial. The scaling 2^{80} is around the magnitude of the lower coefficient in $\mathcal{P}_3(w) \cdot P_{\mathrm{app}}$. More than an educated guess, we have an almost deterministic procedure to find the proper scaling factors to improve our rational reconstructions. This experiment shows that the rational reconstruction is actually easier when the underlying physical problem is taken into account, leading to proper scaling factors.

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